Victor LYUMKIS

METHODS OF ESTIMATION OF ACTIVITY EFFICIENCY TO INCREASE RELIABILITY OF PRODUCTS OUTCOME

The paper is devoted to comparative analysis of efficiency of one or another activity carried out with the purpose of the improvement of needed characteristics of products. In mathematical aspect, the problem can be reduced to the task of the hypothesis testing regarding the activity efficiency to increase products reliability after their finishing. The details of consideration are presented for the Weibull distribution under the II type censoring conditions.

***Keywords**: Weibull distribution, parameters, finishing efficiency, hypothesis testing*

1. Introduction

Increase of efficiency of finishing activities remains an important task in different applications, for example, in aviation, where crucial is to increase after control activity aviation equipment operating time before failure. Often, the task of such kind appears during the development of aircraft, when, due to the finishing done in the construction and technology of each product manufacturing, its quality is changed, as well as the product reliability and aircraft in whole. So, special laboratory and bench tests of various parts of highly reliable products will be the sources of product life span information. In constructing the efficiency criteria we will rely below on such information, although the same criteria may be used for the evaluation of various activities, which efficiency should be evaluated on the basis of the product exploitation results.

Let's consider that initial information is the union of two samplings of observation obtained as a result of tests before and after the controllable activity (CA). Samplings, generally, are censored with censoring of the II type, i.e. they look like

$$x_1, x_2, \dots x_r,$$
$$y_1, y_2, \dots y_s, \quad (1)$$

where r, s - is a number of observations, which appearance causes termination of tests being carried out before and after the CA, n, m – is a corresponding general number of simultaneously tested products [1, 2].

Such performance is rather natural, because due to the duration of fatigue experiments the laboratory tests are continuing only till the r or s failure.

Let us present below a mathematical formulation of the problem of evaluation of activities efficiency for reliability growth. Let us assume, that independent random values (r.v.) X_i, $i=1,...,n$, which represent natural logarithms of time before product failure before the CA, have distribution function (d.f.) with the location and scale parameter like

$$F_{X_i}(u) = \dot{F}_X\left(\frac{u-\theta_{0x}}{\theta_1}\right) \quad (2)$$

and independent r.v. Y_i, $i=1,...,m$, which are natural logarithms of time until the appearance of failure after the CA, have the distribution function like

$$F_{Y_i}(u) = \dot{F}_Y\left(\frac{u-\theta_{0Y}}{\theta_1}\right) \quad (3)$$

Let's notice that the transition to logarithm representation is related to the fact that there is an opportunity in this case to operate with distributions which have the location and scale parameters. Further, we will assume that the CA leaves without change the scale parameter θ_1, but changes the location parameter θ_0 - the value which is closely connected to the mathematical expectation r.v. X_i, $i=1,...,n$. Indeed, if to consider r.v. X and Y, having representation like $X= \theta_{0X}+ \theta_1 \cdot \dot{X}$, $Y= \theta_{0Y}+ \theta_1 \cdot \dot{Y}$, it is clear then that with the invariable θ_1 increase θ_{0X} for $d=(\theta_{0Y}- \theta_{0X})>0$ is equivalent to the increase in the natural scale of mathematical expectation r.v. e^X in $k=e^d$ times. It is obvious then in our case to perform the testing of the hypothesis about the inefficiency of the CA type H: $\theta_{0X} = \theta_{0Y}$, while the alternative if the statement about the increase θ_{0X}, i.e. K: $\theta_{0Y} > \theta_{0X}$ [1].

2. Algorithms for the Weibull distribution

Very often we have to obtain point evaluations of aviation products reliability, which technical parameters have double exponential (Weibull) distribution. Let's consider that independent r.v. X_i, $i=1,...,n$, has f.d. like [3,4]

$$F_{X_i}(u) = 1 - exp(-exp(-exp((u-\theta_{0X})/\theta_1))), \quad (4)$$

and independent r.v. Y_i, $i=1,...,m$, obeys the distribution

$$F_{Y_i}(u) = 1 - exp(-exp(-exp((u - \theta_{0Y})/\theta_1)),\quad (5)$$

According to the observation of the double sampling (1) by the method of maximum likelihood it is possible to evaluate the distribution parameters, while the maximum likelihood function looks like

$$\frac{n!}{(n-r)!}\frac{m!}{(m-r)!}\prod_{i=1}^{r}\frac{1}{\theta_1}f(\frac{x_{(i)}-\theta_{0X}}{\theta_1})\prod_{j=1}^{m}\frac{1}{\theta_1}f(\frac{y_{(j)}-\theta_{0Y}}{\theta_1})*$$

$$*(1-F(\frac{x_{(r)}-\theta_{0X}}{\theta_1}))^{n-r}*(1-F(\frac{y_{(s)}-\theta_{0Y}}{\theta_1}))^{m-s}\quad (6)$$

where $F(u)=1-exp(-exp(-exp(u), f(u)=F'(u)$.

The evaluations of parameters as usual are defined from the equations set

$$\frac{\partial\, ln L}{\partial\theta_1}=0,\quad \frac{\partial\, ln L}{\partial\theta_{0X}}=0,\quad \frac{\partial\, ln L}{\partial\theta_{0Y}}=0.$$

In our case, the equations set looks like

$$\begin{cases} \hat{\theta}_{0X}=\hat{\theta}_1(ln(\frac{1}{r}\sum_{i=1}^{r}v_i)) \\ \hat{\theta}_{0Y}=\hat{\theta}_1(ln(\frac{1}{s}\sum_{i=1}^{m}w_i)) \\ \frac{\sum_{i=1}^{n}v_i x_{(i)}}{\sum_{i=1}^{n}v_i}+s\frac{\sum_{j=1}^{m}w_j y_{(j)}}{\sum_{j=1}^{m}w_j}-(r+s)\hat{\theta}_1-\sum_{i=1}^{r}x_{(i)}-\sum_{j=1}^{s}y_{(j)}=0 \end{cases}\quad (7)$$

where

$$v_i=\begin{cases} exp(x_{(i)}/\hat{\theta}_1), 1\le i\le r, \\ exp(x_{(r)}/\hat{\theta}_1), r<i\le n, \end{cases}$$

$$w_j=\begin{cases} exp(y_{(j)}/\hat{\theta}_1), 1\le j\le s, \\ exp(y_{(s)}/\hat{\theta}_1), s<j\le m, \end{cases}$$

$$x_{(i)}=\begin{cases} x_{(i)}, i\le r, \\ x_{(r)}, i>r, \end{cases}\qquad y_{(j)}=\begin{cases} y_{(j)}, j\le s, \\ y_{(s)}, j>s. \end{cases}$$

Estimator $\hat{\theta}_1$ is determined from the latest equation of the set, and estimators $\hat{\theta}_{0X}$, $\hat{\theta}_{0Y}$ are calculated after putting $\hat{\theta}_1$ in the first and second equations.

For the above formulated task of the hypothesis testing about the inefficiency of the CA type H: $\theta_{0X}=\theta_{0Y}$, against K: $\theta_{0Y}>\theta_{0X}$ let`s use the criterion V, based on statistics

$$V=\frac{\hat{\theta}_{0y}-\hat{\theta}_{0X}}{\hat{\theta}_1}$$

where the above calculated estimators of the method of maximum likelihood are used.

With the known scale parameter θ_1 instead of statistics V it is possible to use statistics

$$U=\frac{\hat{\theta}_{0y}-\hat{\theta}_{0X}}{\theta_1}.$$

The use of regular estimator's properties, in particular, the properties of method of maximum likelihood estimator's allows to notice that the distribution of statistics V, U does not depend on the unknown parameters when the tested hypothesis is true. Really, in this case we have

$$\hat{\theta}_{0Y}=\theta_{oY}+\theta_1\hat{\dot{\theta}}_{0Y}\qquad \hat{\theta}_{0X}=\theta_{oX}+\theta_1\hat{\dot{\theta}}_{0X}\qquad \hat{\theta}_1=\theta_1\hat{\dot{\theta}}_1$$

$$V=\frac{\hat{\theta}_{0y}-\hat{\theta}_{0X}}{\hat{\theta}_1}=\frac{\hat{\dot{\theta}}_{0y}-\hat{\dot{\theta}}_{0X}+\delta}{\hat{\dot{\theta}}_1}$$

$$U=\frac{\hat{\theta}_{0y}-\hat{\theta}_{0X}}{\theta_1}=\hat{\dot{\theta}}_{0y}-\hat{\dot{\theta}}_{0X}+\delta\quad (8)$$

where: $\hat{\dot{\theta}}_{0y}, \hat{\dot{\theta}}_{0X}, \hat{\dot{\theta}}_1$ - estimators of parameters θ_{0X}, θ_{0Y}, θ_1 for observations $\dot{x}_{(1)},...,\dot{x}_{(r)}, \dot{y}_{(1)},...\dot{y}_{(s)}$, having corresponding standard distribution with parameters $\theta_{0X}=\theta_{0Y}=0$, $\theta_1=1$, $\delta=(\theta_{0Y}-\theta_{0X})/\theta_1$ and $\delta=0$, when the hypothesis is true. In properties (8) it is obvious that with the increase $\delta=(\theta_{0Y}-\theta_{0X})/\theta_1$ statistics V, U grow that allows choosing the critical domain where the hypothesis of the following type is denied

$$V>C,\ U>C,$$

where: C is determined from the condition of the first type error limitation or selection of criterion size of the α significance level.

With the truth of alternative from the same properties (8) it is obvious that the distribution of statistics V, U depends only on δ.

Note that situations when time of appearance of r.v. $Y_{(k)}$, $k\geq 1$, becomes extremely prolonged, are possible. Then for the evaluation of the *CA* performance the following statistics are used

$$V_1=\frac{Y_{(1)}-\hat{\theta}_{0X}}{\hat{\theta}_1}\qquad U_k=\frac{Y_{(k)}-\hat{\theta}_{0X}}{\theta_1}\quad (9)$$

i.e. testing of m samples after the CA carrying out is terminated when there is at least one failure $Y_{(1)}$.

Natural generalization of these statistics with $k>1$ are

$$V_K=\frac{Y_{(k)}-\hat{\theta}_{0X}}{\hat{\theta}_1}\qquad U_k=\frac{Y_{(k)}-\hat{\theta}_{0X}}{\theta_1}\quad (10)$$

Let's notice as well, that for the use of criteria (9), (10) it is necessary to evaluate parameter according to one sampling. The corresponding equations set is a particular case of the earlier considered set (7) with substitution of θ_{0X} to θ_0 with $s=m=0$ and it looks like

$$\begin{cases} \dfrac{\sum_{i=1}^{n} v_i x_{(i)}}{\sum_{i=1}^{n} v_i} - r\hat{\theta}_1 - \sum_{i=1}^{r} x_{(i)} = 0 \\ \hat{\theta}_{0X} = \hat{\theta}_1 (ln(\frac{1}{r}\sum_{i=1}^{n} v_i)) \end{cases}$$

where

$$v_i = \begin{cases} exp(x_{(i)} / \hat{\theta}_1), 1 \le i \le r, \\ exp(x_{(r)} / \hat{\theta}_1), r < i \le n, \end{cases} \quad x_{(i)} = \begin{cases} x_{(i)} \ , i \le r, \\ x_{(r)}, i > r. \end{cases}$$

Let's pay attention to the opportunity of approximation getting for the random values distribution V, U, V_k, U_k. Their approximations may be obtained from the use of normal distribution that surely may become a subject of separate consideration.

3. Conclusions and example of algorithm use

Before to formulate conclusions, let's consider the following example. Let time before the appearance of the product failures in the natural scale has the Weibull distribution. We assume that $r = s = 4$, $n = m = 20$ and we have the following observations – numbers of cycles to the fatigue destruction of some products in the natural scale before the CA: $t_{(1)} = 523371$, $t_{(2)} = 577957$, $t_{(3)} = 690711$, $t_{(4)} = 705383$.

After the CA: $t'_{(1)} = 844627$, $t'_{(2)} = 1284670$, $t'_{(3)} = 1427800$, $t'_{(4)} = 1620820$.

After the transition to the natural scale we will obtain observation before and after the CA correspondingly:
$x_{(1)} = 13.163$, $x_{(2)} = 13,627$, $x_{(3)} = 13.445$, $x_{(4)} = 13.466$;
$y_{(1)} = 13.647$, $y_{(2)} = 14.066$, $y_{(3)} = 14.172$, $y_{(4)} = 14.298$.

For the above formulated task of the testing of hypothesis H: $\theta_{0X} = \theta_{0Y}$, against the alternative K: $\theta_{0Y} > \theta_{0X}$ we will use the criterion V_1, based on statistics

$$V_1 = \frac{Y_{(1)} - \hat{\theta}_{0X}}{\hat{\theta}_1}$$

while the parameter estimators for the first sampling have values $\hat{\theta}_1 = 0.148$, $\hat{\theta}_{0X} = 13.174$. Then, the observed value of statistics is $V_1 = 3.196$. After the numerical experiment for the determination of confidence level 0.9 of statistics V_1 with the truth of the hypothesis, we determine the value of the regulatory limit $C = -1.911$. Since $V_1 = 3.196$ exceeds the regulatory limit, the CA should be recognized as the efficient one.

So we can formulate the most significant conclusions:

- for highly reliable products with high cost of experimental research, when the use of the Weibull distribution is needed, the unified method of the activities efficiency evaluator for the reliability growth is offered;
- the efficiency evaluator criteria comes to the hypotheses testing criteria about the efficiency of the products finishing;
- construction of criteria is based on the maximum likelihood method estimators concerning the product failure information before and after the controllable activity;
- the considered algorithms are accompanied by the necessary software aids for the evaluation of the maximum likelihood method parameters and realization of criteria.

4. References

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Dr. sc. ing., Assoc. Professor Victor LYUMKIS
Transport and Telecommunication Institute
Lomonosova 1, Riga, LV-1019, Latvia
tel:+371 7 416731
e-mail: vlyumkis@yahoo.com