1. Introduction

With failures having been removed and design having been perfected during reliability growth tests, the reliability of the product will keep growing [2, 9, 14]. In development of small sample weapon test policy, the product state is usually different at each test stage because changes are made to the design of the product. Therefore the quality and reliability indexes of the product are not a constant because of reliability improvement [11]. The assessment results are not accurate or effective when traditional reliability growth models and non-changeable population hypotheses are used. The probability distribution models for mature product life distributions, such as the exponential distribution, the lognormal distribution, and the Weibull distribution [3-6], do not correctly depict the changing process of product life during the design and test stages. In this paper, we analyze the reliability growth test data based on the monotone model [7], study the data analysis methods of product at various stages, model the trend of dynamic distribution parameters during design and test, and provide a multi-stage Bayesian reliability growth model.

2. Model hypothesis

(1) Assume that a product’s reliability growth process includes h stages. The tests are either fixed time censored (a test is terminated after a pre-determined time duration) or fixed number of failures censored (a test is terminated after a fixed number of failures have been observed). The tests at different stages are independent. The sample sizes for the h stages are denoted by n₁, n₂, · · · , nₕ, the numbers of failures observed are z₁, z₂, · · · , zₕ, and the total test times are τ₁, τ₂, · · · , τₕ.

(2) The product life Tₖ for the kth stage test is assumed to follow the exponential distribution with probability distribution function (PDF) of f(t) = λₑe⁻ₑt, t > 0. If the mission time is Tₑ, the product reliability of the kth stage product for the mission is:

Rₖ = Pr(Tₑ ≥ Tₖ) = e⁻ₑₖ, k = 1, · · · , h (1)

(3) The product state is assumed to be steady within each of the k(1 ≤ k ≤ h) stages. The product reliability increases from stage to stage because of removing defects, thus:

0 ≤ R₁ ≤ R₂ ≤ · · · ≤ Rₙ₁ ≤ Rₙ ≤ 1 (2)

Obviously, the assumption in equation (2) is equivalent to the following monotone model when the exponential distribution is used:

∞ ≥ λ₁ ≥ λ₂ ≥ · · · ≥ λₕ > 0 (3)

Therefore, exponential distribution reliability growth is modeled by the monotone model, i.e. expression (3).

3. Statistical method of reliability data

The monotone model is easy to use when there is little reliability data [12]. Therefore, this paper deals with product reliability data by the monotone model in order to get point estimation and confidence lower limit of reliability index. The confidence limit is used to determine values of hyper-parameters of Bayesian prior distribution.

For stage k, the prior distribution of the failure rate λₑ of the exponential distribution is expressed in terms of a gamma distribution. The form of the gamma distribution with shape parameter aₖ and scale parameter bₖ is:
The likelihood function of the fixed time censored test with replacement \((t_r, z_r)\) is expressed as following.

\[
L(\lambda_i) = \frac{(t_r \lambda_i)^{z_r} \exp(-\lambda_i t_r)}{z_r!}
\]

According to the Bayes formula, the posterior distribution can be described by the following expression.

\[
\pi(\lambda_j|D_k) \propto \int_{a_k}^{b_k} \pi(\lambda_j|a_k, b_k) L(\lambda_i) d\lambda_i = \frac{\int_{a_k}^{b_k} \pi(\lambda_j|a_k, b_k) L(\lambda_i) d\lambda_i}{\int_{a_k}^{b_k} L(\lambda_i) d\lambda_i}
\]

\[
= \frac{(b_j + t_r \gamma)^{z_r+1}}{z_r!} \cdot \lambda_j^{z_r+1} \exp(-\lambda_j (b_j + t_r))
\]

where \(D_k\) denotes the test result of the stage \(k\) test as described in assumption (1), \(\pi(\lambda_j|D_k)\) is the posterior distribution of \(\lambda_j\) obtained from the prior \(\pi(\lambda_j|a_k, b_k)\) and the likelihood function obtained from the test result \(D_k\). Thus, the upper confidence limit \(\lambda_{U(k)}\) of the failure rate satisfies the following expression (10).

\[
\lambda_{U(k)} = \int_{a_k}^{b_k} \lambda_j \pi(\lambda_j|D_k) d\lambda_j = \gamma
\]

where \(\gamma\) is the confidence level. The point estimate of the failure rate is described by the following expression.

\[
\hat{\lambda}(k) = \int_{a_k}^{b_k} \lambda_j \pi(\lambda_j|D_k) d\lambda_j
\]

After the \(k^{th}\) test the lower confidence limit of MTBF and the point estimate of MTBF are given by formula (12) and formula (13), respectively.

\[
MTBF_\gamma(k) = \frac{1}{\hat{\lambda}(k)}
\]

\[
MTBF_{\text{test}}(k) = \frac{1}{\lambda(k)}
\]

4. Reliability growth test

In order to test whether the product reliability is growing or not, we need to check if the test data confirm the monotone relationship, i.e. expression (2) or (3), via hypothesis testing [14].

For stage \(i\) and \(i+1\), calculate the following test statistics based on the test data obtained following assumption (1):

\[
F^* = \begin{cases} 
\frac{\tau_{i+1} z_i}{\tau_{i+1} (2 z_i + 1)}, & \text{for fixed number of failures censored test} \\
\frac{\tau_{i+1} (2 z_i + 1)}{\tau_i z_i + 1}, & \text{for fixed time censored test}
\end{cases}
\]

Then the \(F^*\) follows the \(F\) distribution with degrees of freedom of \((2 z_i, 2 \tau_i)\) for the fixed time censored case, and with degrees of freedom of \((2 z_{i+1}, 2 \tau_{i+1})\) for the fixed number of failures censored case. If

\[
F^* \geq F_{1-\alpha}(2 z_i, 2 \tau_i)
\]

we conclude that reliability grows from stage \(i\) to stage \(i+1\). The condition (15) is not satisfied, we conclude that the data from these two stages are not different. In this case, we merge the data from these two stages and conduct a reliability growth test with the data from the next stage, that is, stage \(i+2\). In condition (15), \(F_{1-\alpha}(2 z_i, 2 \tau_i)\) is the upper 100(1-\(\alpha\))% point of the \(F\) distribution with degrees of freedom of \((2 z_i, 2 \tau_i)\) for the fixed time censored test, \(\alpha\) is the significance level (\(\alpha\) is usually chosen to be 0.2). If we are convinced that the product reliability has been improved, we can set a tighter requirement by choosing the value of \(\alpha\) to be 0.3, 0.4 or more. However, the idea of merging the data from two stages that are not significantly different and compare them with the next stage is difficult to operate in practice. The reason is that if design flaws have been removed during product development, then the data from the improved design cannot be merged with the data from the design without the improvement. To solve this problem, we apply the inheritance factor method [13] in this merge process. Furthermore, the inheritance factor must be determined before merging data from adjacent stages and then reliability growth test can be conducted. We discuss the calculation of the inheritance factor in the next section.

5. Inheritance factor calculation

Expert assessment method and goodness of fit test are affected by many subjective factors. In order to decrease the subjectivity influence, this paper determines the inheritance factor by an information fusion method, i.e., Kullback method [8].

The definition of Kullback information between cumulative distribution function \(P\) and cumulative distribution function \(Q\) in the metric space \((\mathcal{M},\mathcal{F})\) is described by the following expression:

\[
\Omega(P, Q) = \int_{\mathcal{M}} P(x) \log \frac{P(x)}{Q(x)} dx
\]
where $\Omega$ is the definition domain, $F$ is $\sigma$ algebra in $\Omega$, $FQ(\cdot)$ is an expectation relative to $Q$. $\frac{dQ}{dP}$ is Radon-Nikodym derivative of $Q$ about $P$. When $Q$ is much smaller than $P$, it means that $Q$ is absolutely continuous about $P$. If $\Omega$ is a real space, then $P$ and $Q$ can be described by the following expression [10]:

$$P(dx) = p(x)dx \quad Q(dx) = q(x)dx$$

(17)

In this case, the definition of the Kullback information is equivalent to formula (18).

$$I(Q:P) = I(q:p) = \int \left[ \ln \frac{q(x)}{p(x)} \right] p(x)dx$$

(18)

If $\Omega$ is a finite discrete space, i.e. $\Omega = \{x_1, x_2, \ldots, x_n\}$, following $P(x_i) = p_i$, $Q(x_i) = q_i$, $i = 1, 2, \ldots, m$, the Kullback information between $P$ and $Q$ is [10]:

$$I(Q:P) = \sum_{i=1}^{m} q_i \ln \left[ \frac{q_i}{p_i} \right]$$

(19)

Furthermore, $I(Q:P) \geq 0$ and the necessary and sufficient condition is $P = Q$ when the expression is equal to zero.

Since Kullback information is the measurement of the diversity of two distributions, the more similar two distributions are, the larger the Kullback information is. Furthermore, while the range of the inheritance factor is $[0, 1]$, the Kullback information is not always in the $[0, 1]$ interval. So the monotone function described in expression (20) is used to transform the Kullback information into a function whose range is $[0, 1]$:

$$\rho = 1 - \frac{|I(Q:P)|}{|I(Q:P)| + 1}$$

(20)

where $\rho$ is the inheritance factor.

6. Reliability growth test plan formulation

Reliability growth test will be done when the design is capable of performing the intended function so that the designer can find design flaws and improve design. Usually it is impossible to fulfill the specified reliability requirement in the initial development stage of a large complex system. Therefore, it is important to do reliability growth test in the initial prototype development in order to remove the flaws of design, manufacture and operation and improve product reliability. Thus, reliability growth test plan formulation is important for new system and product development.

6.1. How to determine parameters of reliability growth

In order to develop a reliability growth test plan, the parameters of reliability growth must be determined according to the actual situation, which usually include growth objective, reliability growth rate and initial value of growth. The objective value can be easily determined by the reliability requirement of product.

The improvement effect and the total test time can be affected by reliability growth rate. To choose a proper growth rate, we must fully consider the reliability growth of similar products and the assumed improvement measures. Basically the growth rate can be determined by reliability data when there is enough data [12]. If reliability data of product is available, the estimation of the failure rate is $\lambda$, at time $t$.

Assuming that the reliability growth of product follows Duane model, the estimation of growth rate $m_i$ is:

$$m_i = 1 - \frac{\lambda_{T_i}}{N_i}$$

(21)

where $N_i$ is the total number of failures observed up to stage $i$, $N_i = \sum_{j=1}^{i} z_j$. Therefore the growth rate $m$ at the end of stage $h$ is as given in expression (22).

$$m = \frac{1}{h} \sum_{i=1}^{h} m_i = 1 - \frac{1}{h} \sum_{i=1}^{h} \frac{\lambda_{T_i}}{N_i}$$

(22)

After determining the growth rate $m$, the initial value of reliability growth parameters, $T_i$ and $M_i$ can be given by expression (23):

$$T_i = t_i$$

$$M_i = -(1-m)MTBF_{\kappa_{n}}(T_i)$$

(23)

where, $T_{i}=t_{i}$ is the accumulated test time of the product to the end of stage $h$, $M_i$ is the accumulated MTBF, $MTBF_{\kappa_{n}}(T_i)$ is a point estimate of the mean time between failures at time $t_i$.

6.2. How to determine total time of reliability growth test

The total test time directly affects reliability test resources. Thus, it must be controlled carefully. We can use the following expression [9, 14].

$$T \geq T_i \left( \frac{(1-m)M_{\kappa_n}}{M_{i}} \right)^{\frac{1}{m}}$$

(24)

If parameters of $(T_i, M_i)$, $M$ and $M_{\kappa_n}$ are known, the total time is fixed.

7. Example and conclusion

Assume that the development of a product includes principle prototype, initial prototype and formal prototype. The product will undergo multiple stage tests in the development process. It is necessary to make a reliability growth plan in order to improve test efficiency. During reliability growth tests, 4 failures were observed in the principle prototype. After failure cause analysis and design improvement the mission time of product was 10 hours and the test was fixed time censored. Tab. 1 shows the data obtained from each test stage.

<table>
<thead>
<tr>
<th>Test stage</th>
<th>Number of failures</th>
<th>Total test time (hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The first stage</td>
<td>4</td>
<td>42.9</td>
</tr>
<tr>
<td>The second stage</td>
<td>2</td>
<td>45.4</td>
</tr>
<tr>
<td>The third stage</td>
<td>1</td>
<td>62.5</td>
</tr>
</tbody>
</table>

The reliability growth effect of different stages has been tested by the method provided in Section 3 of this paper. When $\alpha = 0.2$ the test statistic between stage 1 and stage 2 was equal to 1.9049, i.e., $F^{*}=1.9049$, larger than the critical value of 1.8455. The test statistic between stage 2 and stage 3 was 2.2894, i.e., $F^{*}=2.2894$, also larger than the critical value of 2.2530. Thus the
reliability of the product obviously improved in the development stage. Furthermore the point estimation of the MTBF can be obtained by expressions (11) and (13), which is 103h.

In summary, this paper provided a statistical analysis method for multiple stages test data based on the monotone model, described the change event of dynamic distribution parameters during testing, modeled the Bayesian reliability growth model of multiple stages exponential distribution product, and considered the influence of non-Homogeneous population information. The result of an example demonstrated the usefulness of the proposed method for reliability growth assessment.

8. References