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MAINTENANCE OPTIMIZATION FOR SYSTEMS WITH DEPENDENT COMPETING RISKS USING A COPULA FUNCTION

OPTYMALIZACJA EKSPLOATACJI DLA SYSTEMÓW Z ZALEŻNYMI ZAGROŻENIAMI KONKURUJĄCYMI PRZY WYKORZYSTANIU FUNKCJI KOPUŁY

This paper develops a joint copula reliability model for systems subjected to dependent competing risks caused by two degradation processes and random shocks. The two degradation processes follow gamma processes and the random shocks follow a non-homogeneous Poisson process (NHPP). Their interdependence relationship is modeled by a copula function, which is determined by a two-stage method based on simulated data. It is shown that the proposed model can provide more precise results than the model without considering the dependent relationship. Through the proposed reliability model, two maintenance models are studied and compared. It is found that the inspection cost has significant effects on the choosing of maintenance policy.

***Keywords**: dependent competing risks, copula function, simulated data, degradation, random shocks, maintenance optimization.*

W niniejszej pracy opracowano wspólny model niezawodności z użyciem kopuły dla systemów poddawanych zależnym zagrożeniom konkurującym powodowanym przez dwa procesy degradacji i zaburzenia losowe. Owe dwa procesy degradacji reprezentują typ procesu gamma, podczas gdy zaburzenia losowe są typem niejednorodnego procesu Poissona (non-homogeneous Poisson process - NHPP). Ich związek wzajemnej zależności modelowany jest przy użyciu funkcji kopuły, która jest wyznaczana na podstawie dwuetapowej metody opartej o dane symulowane. Wykazano, iż proponowany model może zapewnić bardziej precyzyjne wyniki niż model, w którym nie ujęto związku zależności. W oparciu o proponowany model niezawodności, badane i porównywane są dwa modele eksploatacji. Stwierdzono, iż koszt przeglądu ma duży wpływ na wybór polityki eksploatacyjnej.

***Słowa kluczowe**: zależne ryzyka konkurujące, funkcja kopuły, dane symulowane, degradacja, zaburzenia losowe, optymalizacja eksploatacji.*

1. Introduction

Competing risks are quite common situations in industry for systems or components which can be subjected to more than one causes of failure at the same time and fail due to one of them [17,19]. Therefore, it is beneficial to consider the competing risks for the maintenance scheduling.

Many studies treat the competing risks as independent failure processes. Lehman [17] investigated a class of degradation-threshold-shock models in which the failure is caused by the competing risks of degradation and trauma. Bocchetti et al. [2] proposed a model to describe the competing risks caused by wear degradation and thermal cracking for the cylinder liners in marine diesel engine. Due to the complex features of lifetime data, Jiang [13] developed a competing risk model involving a geometric distribution and an exponential Poisson distribution to model bus-motor failure data. Li and Pham [18] presented an inspection-maintenance model for systems subjected to two degradation processes and random shocks. Zhu et al. [34] presented a maintenance model that maximizes the unit availability by determining the degradation threshold level and the time to perform preventive maintenance (PM). Kharoufeh et al. [14] derived the system lifetime distribution and the limiting average availability for a periodically inspected system, which is subjected to degradation and random shocks modulated by a homogeneous Poisson process. Wang et al. [30] studied the impact of shocks on the product and found that the shocks had a significant impact on the product reliability.

The assumption of s-independence between competing risks may cause underestimation or overestimation of the system reliability and has substantial impacts on maintenance optimization [3]. Therefore, it is essential to take account of the dependent relationship in order to model the reliability more accurately and make more appropriate maintenance strategy.

Some recent papers have incorporated the dependent relationship into the reliability modeling process. Su and Zhang [26] studied the reliability assessment for GaAs lasers based on competing risk model. The results show that the dependence between the traumatic failure and degradation has a great influence on the accuracy of reliability assessment. Considering the dependency between wear failure and shock failure, Jiang and Coit [12] developed reliability models with two classes of shock processes and a linear degradation process. The arrival of each shock impacts both the soft failure process and the hard failure process. Pan and Balakrishnan [21] proposed to use a bivariate Brinbaun-Saunders distribution to describe the dependent relationship between the two gamma degradation processes and developed an inferential method for the corresponding model parameters. Singpurwalla [25] proposed a general framework for an appreciation of competing risks and degradation involving interdependent stochastic processes under the notion of a hazard potential. Pan and Zhao [22] treated the problem of accelerated failure with competing causes of a degradation failure mode and multiple traumatic failure modes. Abbring and van den Berg [1] studied the dependent competing risks models with a mixed proportional hazard for each risk. Wang and Coit [27] proposed a general modeling and analysis approach for reliability prediction based on multiple degradation measures and illustrated the approach with multivariate Normal distributions.

There has also been a growing interest in considering the maintenance optimization with dependent competing risks in recent years. Klutke and Yang [15] studied the average availability of maintained systems subject to shocks and graceful degradation with hidden failures. Huynh et al. [10] developed a dependent competing risk model by assuming the arrival rate of shocks as a function of the degradation level, and proved the value of condition monitoring to the maintenance decision-making. Later Huynh et al. [11] developed age-based maintenance strategies with minimal repairs for systems based on the same competing risk model. Wang and Pham [28] studied a multi-objective optimization problem of imperfect preventive maintenance policy for a single-unit system subjected to the dependent competing risks, by simultaneously maximizing the system asymptotic availability and minimizing the system cost rate. It is assumed that fatal shocks will cause the system to fail immediately, whereas nonfatal shocks will increase the system degradation level by a certain cumulative shock amount. In order to give a more explicit dependent relationship, Chen [7] used the degradation level as a variable of the arrival rate function of the fatal shock, and an inspection/replacement policy is discussed based on the proposed model. Castro [4] developed a dependent relationship for two competing failure modes in which the non-maintainable failure number affects the maintainable failure rate. The optimal number of PMs and the interval between successive PMs are determined with the objective of minimizing the expected cost rate. Zequeira and Bérenguer [32] studied the imperfect maintenance policies with the consideration of two competing failure modes, where the hazard rate of the maintainable failure mode depends on the hazard rate of the non-maintainable failure mode. Deloux et al. [9] considered a system with two failure mechanisms due to an excessive deterioration level and a shock. The optimal maintenance strategy is studied in an approach which combines statistical process control and condition-based maintenance. Peng et al. [23] presented a preventive maintenance policy for systems subjected to multiple competing failures where the external random shocks contribute to the internal degradation.

Previous researches have mainly investigated the dependence relationships among degradation processes by multivariate normal distribution, and modeled the failure rate with covariates etc. Though the system reliability functions can be deduced directly, these approaches are insufficient to cope with the complexity of the modern system in reality [29, 33].

Copula is a powerful tool to model the dependence of random variables, and the copula based models allow for flexible specification of the dependence structure between competing random variables [3, 24]. Zhou [33] proposed a bivariate degradation modeling framework based on gamma processes and copula function is used to describe the dependence between performance characteristics. Wang and Pham [29] developed a flexible s-dependent competing risk model to describe the dependence between random shocks and the degradation process by employing time-varying copulas. Lo and Wilke [20] extended the copula graphic estimator to model multiple dependent competing risks and applied the model to the unemployment duration data from Germany. However, copula function has seldom been applied to model the dependence structure in maintenance optimization.

In practice, systems are usually subjected to competing risks involving both degradation and shocks, as investigated by many researchers [6, 10, 12, 15 and 30]. In this paper, a system suffering dependent competing risks caused by two degradation processes and random shocks is studied. With the dependence structure modeled by copula function, a joint reliability function is developed based on the simulated data and the maintenance optimization is investigated.

The remaining paper is organized as follows. Section 2 investigates the system failure process and deduces the marginal reliability function for the system suffering two degradation failure processes and random shocks. Section 3 develops the system reliability model based on a copula function and provides a parameter estimation procedure based on simulated data. Section 4 presents two maintenance models based on the joint copula reliability function. In Section 5, a numerical example is presented to illustrate the procedure to determine the joint reliability function and investigate the maintenance optimization for the two maintenance policies.

2. Dependent competing risks

Consider a system subjected to competing risks due to two degradation processes and random shocks. The two degradation processes have a dependent relationship with each other as each shock causes a sudden increment jump to both degradation processes simultaneously. The system fails if the cumulative deterioration of any degradation process exceeds a certain critical failure threshold.

2.1. Degradation processes without random shocks

Gamma processes have been extensively adopted to describe the gradual degradation phenomena e.g. corrosion [16], crack growth [5]. Let $X_i(t)$, ($i=1,2$) denote the accumulated deterioration for the i th degradation process at time t, where the initial state of the system is perfect with $X_i(0)=0$. Assume that $\{X_i(t), t\geq 0\}$, ($i=1,2$) is a stationary gamma process where $X_i(t)-X_i(s)$ is gamma distributed for all $0\leq s<t$. Without considering the influences of the shock process, $X_i(t)-X_i(s)$, $0\leq s<t$ has a gamma probability density function (pdf) with shape parameter $\alpha_i(t-s)>0$ and scale parameter $\beta_i>0$:

$$f_{\alpha_i(t-s),\beta_i}(x_i)=\frac{\beta_i^{\alpha_i(t-s)}x_i^{\alpha_i(t-s)-1}e^{-\beta_i x}}{\Gamma(\alpha_i(t-s))}I_{\{x_i\geq 0\}}, \quad (1)$$

where $\Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}du$ is the gamma function. $I_{\{x_i\geq 0\}}=1$ if $x_i\geq 0$ and $I_{\{x_i\geq 0\}}=0$ otherwise.

The average deterioration rate is $u_i=\alpha_i/\beta_i$, and its variance is $\sigma_i^2=\alpha_i/\beta_i^2$. Though the constant deterioration rate may be unsuitable for the realistic degradation process, a monotonic transformation of the time scale can make the real deterioration rate constant [31]. With the choice of α_i and β_i, such a process can be very flexible to model various deterioration behaviors of the system.

The stochastic process $\{X_i(t), t\geq 0\}$ is time continuous and monotonically increasing, and the system fails once $X_i(t)$ exceeds a predetermined failure threshold L_i. Though the system may be still functioning after crossing the failure threshold, it cannot perform its function as required and is regarded as "failed" for economical or security reasons. The time to failure (TTF) of the i th degradation process can be expressed as $TL_i=\inf\{t \mid X_i(t)\geq L_i\}$, and its cumulative distribution function (cdf) can be obtained as:

$$F_{TL_i}(t) = P(TL_i \le t) = P(X_i(t) \ge L_i) = \int_{L_i}^{\infty} f_{\alpha_i t, \beta_i}(x)dx = \frac{\Gamma(\alpha_i t, \beta_i L_i)}{\Gamma(\alpha_i t)}, \quad (2)$$

where $\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt$.

The pdf for TTF of the i th degradation process is

$$f_{TL_i}(t) = \frac{\partial}{\partial t} F_{TL_i}(t) = \frac{\alpha_i}{\Gamma(\alpha_i t)} \int_{L_i \beta_i}^{\infty} (\ln(u) - \psi(\alpha_i t)) u^{\alpha_i t - 1} e^{-u} du, \quad (3)$$

where $\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)} = \frac{\partial}{\partial a} \ln \Gamma(a)$ is called the digamma function.

The reliability function corresponding to the i th degradation process is

$$R_{TL_i}(t) = 1 - F_{TL_i}(t) = 1 - \frac{\Gamma(\alpha_i t, \beta_i L_i)}{\Gamma(\alpha_i t)}. \quad (4)$$

2.2. Shock process

Shocks may be generated internally within components or introduced externally from the environment outside. Most shocks are harmful to the system operation, and can reduce the system residual useful life. In this paper, a cumulative shock model is employed to describe the shock process. The probabilities for the shock damages to occur in different time intervals are assumed to be independent.

The log-linear process (LLP) is very flexible and has been widely used to describe the occurrence of random events, such as the wear of cylinder liner [2]. Here the shock process is described by the LLP, and the random shocks are assumed to occur in a non-homogeneous Poisson process (NHPP) with intensity function

$$\lambda(t) = r e^{ct}, \ r \in (0, \infty), \ c \in (-\infty, +\infty). \quad (5)$$

Let $N(t)$ denote the number of shocks until time *t*, then the expected number of shocks until time *t*, denoted by $W(t)$, is given by

$$W(t) = E[N(t)] = \int_0^t r e^{cs} ds = \begin{cases} \frac{r}{c}(e^{ct} - 1), & c \ne 0 \\ rt, & c = 0 \end{cases}. \quad (6)$$

Further, the probability distribution of $N(t)$ is

$$P(N(t) = n) = \frac{(W(t))^n}{n!} e^{-W(t)}. \quad (7)$$

The amount of damage caused by the k th shock to the i th degradation process is denoted by S_{ik} and $S_{ik} \sim N(\mu_i, \sigma_i^2)$. Furthermore, the accumulated shock damages to the i th degradation process until time t is expressed as $Z_i(t) = \sum_{k=1}^{N(t)} S_{ik}$.

Consider $G(l) = P(S_{ik} \le l)$ as the cdf for all S_{ik} . The cdf for the accumulative shock damage to the i th degradation process incurred by the shock process is

$$\begin{aligned} P(Z_i(t) \le z) &= P(\sum_{k=1}^{N(t)} S_{ik} \le z) \\ &= P(N(t) = 0) + \sum_{j=1}^{\infty} G^{(j)}(z) P(N(t) = j) = e^{-W(t)} + \sum_{j=1}^{\infty} \Phi(\frac{z - j\mu_i}{\sqrt{j\sigma_i^2}}) \frac{(W(t))^j}{j!} e^{-W(t)}, \end{aligned} \quad (8)$$

where $G^{(j)}(z)$ is the j-fold convolution with itself.

2.3. Degradation processes with random shocks

Section 2.1 investigated the reliability of the system subjected to the degradation process, without considering the influences induced by the shock process. In practical applications, the random shocks may exist and have impacts on the degradation processes. [29]

In this paper, the random shocks will induce a sudden increment to the degradation process. Considering the dependent relationship of degradation processes and random shocks, the i th degradation process state $Y_i(t)$ includes two parts: the wear caused by the system aging and the sudden increments induced by the random shocks. The i th degradation at time *t* can be expressed as $Y_i(t) = X_i(t) + Z_i(t)$. Denote the TTF for the i th degradation by T_i . The reliability function for the i th degradation process with random shock damages is given by

$$\begin{aligned} R_i(t) &= P(T_i > t) = P(Y_i(t) < L_i) = P(X_i(t) + Z_i(t) < L_i) \\ &= \sum_{k=0}^{\infty} P(X_i(t) + Z_i(t) < L_i \mid N(t) = k) P(N(t) = k) \\ &= P(N(t) = 0) P(X_i(t) < L_i) + \sum_{k=1}^{\infty} P(N(t) = k) \int_0^{L_i} P(X_i(t) + z < L_i) dG^{(k)}(z) \\ &= e^{-W(t)} (1 - \frac{\Gamma(\alpha_i t, \beta_i L_i)}{\Gamma(\alpha_i t)}) + \sum_{k=1}^{\infty} \frac{(W(t))^k}{k!} e^{-W(t)} \int_0^{L_i} (1 - \frac{\Gamma(\alpha_i t, \beta_i (L_i - z))}{\Gamma(\alpha_i t)}) dG^{(k)}(z). \end{aligned} \quad (9)$$

The pdf of TTF for the ith degradation process with random shocks can be expressed as

$$f_i(t) = -\frac{dR_i(t)}{dt}. \quad (10)$$

3. System reliability analysis

The system failure occurs if any of the degradation processes $Y_i(t)$ reaches the failure threshold L_i . Therefore, the system reliability at time t is

$$R(t) = P(Y_1(t) < L_1, Y_2(t) < L_2) = P(X_1(t) + Z_1(t) < L_1, X_2(t) + Z_2(t) < L_2) \quad (11)$$

If the two degradation processes are independent, the system reliability function can be written as

$$R(t) = R_1(t) R_2(t). \quad (12)$$

However, Eq. (12) is unable to provide the accurate system reliability estimation for our case, as there is dependency between the two degradation processes due to the random shocks. It is difficult to calculate $R(t)$ by Eq. (11) directly, so we need to find another way to predict the reliability of the system subject to dependent competing failures.

3.1. A Copula approach

A Copula function is a powerful tool to model the dependence structure of the competing failure processes. One advantage of the copula function is that the joint reliability function can be modeled directly through the univariate marginal reliability functions of the individual failure processes, (i.e. $F_1(t)$, $F_2(t)$) and the copula has no constraints on the univariate marginal distribution.

The cdf of TTF for the two degradation processes can be expressed as $F_i(t) = 1 - R_i(t)$ ($i = 1, 2$), and the joint cdf of T_1 and T_2 is denoted by $H(t_1, t_2)$. According to Sklar's theorem, there exists a unique copula C such that

$$P(T_1 \le t_1, T_2 \le t_2) = H(t_1, t_2) = C(F_1(t_1), F_2(t_2), \Theta), \quad (13)$$

where Θ is the parameter vector of the copula function.

Meanwhile, the joint reliability function of the system with t_1 and t_2 can be expressed as

$$\bar{H}(t_1, t_2) = P(T_1 > t_1, T_2 > t_2) . \quad (14)$$

Because $R_1(t)$ and $R_2(t)$ are decreasing functions, the system reliability at time t ($t_1 = t_2 = t$) can be expressed with the survival copula function as [8, 24]

$$\begin{aligned} R(t) &= \bar{H}(t_1, t_2)|_{t_1=t_2=t} \\ &= R_1(t_1) + R_2(t_2) - 1 + C(F_1(t_1), F_2(t_2), \Theta)|_{t_1=t_2=t} \\ &= R_1(t) + R_2(t) - 1 + C(F_1(t), F_2(t), \Theta). \end{aligned} \quad (15)$$

There is another approach to construct the system reliability with a copula function, as shown in [29]. The joint reliability function can be directly modeled by a copula function and can be written as

$$R(t) = \bar{H}(t_1, t_2)|_{t_1=t_2=t} = C(R_1(t_1), R_2(t_2), \Theta)|_{t_1=t_2=t} = C(R_1(t), R_2(t), \Theta) . \quad (16)$$

The results of Eq. (15) and (16) may be different, and we will compare the two approaches in Section 5.

3.2. Parameter estimation

Assume that the parameters of the marginal reliability functions for the degradation processes are already given. In order to predict the system reliability, we need to estimate the copula parameters based on the known marginal distributions. The pdf of the joint distribution $\bar{H}(t_1, t_2)$ can be denoted as $f(t)$ as $t_1 = t_2 = t$. Further, we can obtain $f(t)$ from Eq. (15) as

$$\begin{aligned} f(t) &= f(t_1, t_2)|_{t_1=t_2=t} \\ &= -\frac{\partial^2}{\partial t_1 \partial t_2}(R_1(t_1) + R_2(t_2) - 1 + C(F_1(t_1), F_2(t_2), \Theta))|_{t_1=t_2=t} \\ &= f_1(t_1) + f_2(t_2) - c(F_1(t_1), F_2(t_2), \Theta) f_1(t_1) f_2(t_2)|_{t_1=t_2=t} \\ &= f_1(t) + f_2(t) - c(F_1(t), F_2(t), \Theta) f_1(t) f_2(t), \end{aligned} \quad (17)$$

where $c(F_1(t_1), F_2(t_2), \Theta) = \frac{\partial^2}{\partial F_1(t_1) \partial F_2(t_2)} C(F_1(t_1), F_2(t_2), \Theta)$ is the copula density function.

Similarly, $f(t)$ for Eq. (16) is given as

$$f(t) = f(t_1, t_2)|_{t_1=t_2=t} = -c(R_1(t_1), R_2(t_2), \Theta) f_1(t_1) f_2(t_2)|_{t_1=t_2=t} , \quad (18)$$

where $c(R_1(t_1), R_2(t_2), \Theta) = \frac{\partial^2}{\partial R_1(t_1) \partial R_2(t_2)} C(R_1(t_1), R_2(t_2), \Theta)$.

In this paper, the simulated data are used to estimate the parameters of the copula function and validate the effectiveness of the copula method. The proposed method can be divided into two stages.

In the first stage, we need to simulate the competing failure processes to obtain the system marginal reliability sample with the underlying dependent relationship between the degradation processes and the shock process at discrete times. The procedures are described as follows:

- Compute the degradation increment $X_i(t)$ $(i = 1, 2)$ of each degradation process at $t = m\Delta t$ (m =1,2,…), where Δt is the time step for the degradation process simulation.
- Generate the shock arrival times following NHPP $\{t_1, t_2, ..., t_n\}$, ($t_n \le t$) and the corresponding shock damages to each degradation process $\{s_{i1}, s_{i2}, ..., s_{in}\}$.
- Compute the accumulated shock damage at time t as $Z_i(t) = \sum_{k=1}^{n} s_{ik}$ for each degradation process.
- Compute the system reliability $\hat{R}(t) = \sum_{j=Num} I_{\{X_i(t_j)+Z_i(t_j) \le L_i\}} / Num$,where I is an indicator function. $I = 1$, if $X_i(t) + Z_i(t) \ge L_i$ and $I = 0$ otherwise. Num is the total number of simulations.

In the second stage, the Maximum likelihood estimator (MLE) is used to estimate the copula function parameters based on the simulated marginal reliability sample. Below are the procedures:

Consider N_1 simulated results for the degradation processes, which are denoted by $\{F_1(t_j), F_2(t_j)\}_{j=1,\cdots N_1}$. With Eq. (17) and (18), the log-likelihood function for the bivariate copula can be expressed respectively as

$$\ln L(\Theta) = \sum_{j=1}^{N_1} \ln c(F_1(t_j), F_2(t_j), \Theta) , \quad (19)$$

$$\ln L(\Theta) = \sum_{j=1}^{N_1} \ln c(R_1(t_j), R_2(t_j), \Theta) . \quad (20)$$

Using MLE, the copula parameters can be estimated as

$$\hat{\Theta} = ArgMax\{\ln L(\Theta)\} . \quad (21)$$

4. Maintenance models

This section presents two kinds of maintenance policies based on the joint copula reliability function for a non-repairable system. The first policy is a periodic inspection/replacement policy and the decision variable for maintenance decision maker is the inspection interval. The second policy is an age-based maintenance policy and the decision variable is the replacement age to be specified. For both maintenance policies, the objective is to minimize the average maintenance cost rate in long run.

4.1. Periodic inspection/replacement policy

Due to cost reasons and other practical issues, the system is inspected at a periodic interval τ. The inspection is perfect and instantaneous with a cost C_I incurred. When any of the two degradation processes with the underlying shock damages exceeds the pre-set threshold, the system is deemed as failed though it still runs until the failure is identified at the next inspection. In case when a system failure is identified at an inspection, it is replaced instantly with a new one and the replacement time is negligible. The replacement can be seen as a renewal.

Denote the accumulative maintenance cost until time t as $C(t)$. According to the renewal theory, we have

$$\lim_{t\to\infty}\frac{C(t)}{t}=\frac{E[CR]}{E[TR]}, \tag{22}$$

where $E[CR]$ is the expected total maintenance cost in a renewal cycle, $E[TR]$ is the expected length of a renewal cycle.

The maintenance costs in a renewal cycle are composed of inspection cost, replacement cost and the delay time cost during system failure period. The delay time cost is incurred by the loss of system performance during the system failure period. The expected total cost in a renewal cycle can be expressed as

$$E[CR]=C_I E[N_I]+C_D E[\xi]+C_R, \tag{23}$$

where C_I is the cost associated with each inspection, C_D is the delay time cost rate for the system failure duration, C_R is the replacement cost after the system failure, $E[N_I]$ is the expected inspection number in a renewal cycle, $E[\xi]$ is the expected time that the system spends in failed state in a renewal cycle.

Denote the failure time of the system as $T=\min(T_1,T_2)$. If there are i inspections in a renewal cycle, then we have $\{N_I=i\}=\{(i-1)\tau<T\le i\tau\}$. Therefore, the expected number of inspections in a renewal cycle is given as

$$E[N_I]=\sum_{i=1}^{\infty}iP(N_I=i)=\sum_{i=1}^{\infty}iP((i-1)\tau<T\le i\tau)=\sum_{i=1}^{\infty}i(F_T(i\tau)-F_T((i-1)\tau)), \tag{24}$$

where $F_T(t)$ is the cdf of the TTF of the system, which can be calculated by $1-R(t)$ based on Eq. (15) and (16).

If the system is identified as failed at the i th inspection, then the delay time is $\xi=i\tau-T$. Therefore, the expected delay time of the system in a renewal cycle is

$$E[\xi]=\sum_{i=1}^{\infty}E[\xi\mid N_I=i]P(N_I=i)=\sum_{i=1}^{\infty}\left(\int_{(i-1)\tau}^{i\tau}(i\tau-t)dF_T(t)\right). \tag{25}$$

The expected length of a renewal cycle can be expressed as

$$E[TR]=\sum_{i=1}^{\infty}i\tau P(N_I=i)=\sum_{i=1}^{\infty}i\tau(F_T(i\tau)-F_T((i-1)\tau)). \tag{26}$$

Based on Eq. (22)-(26), the average maintenance cost rate in long run is given as a function of τ as

$$AVC(\tau)=\frac{C_I\sum_{i=1}^{\infty}i(F_T(i\tau)-F_T((i-1)\tau))+C_D\sum_{i=1}^{\infty}\left(\int_{(i-1)\tau}^{i\tau}(i\tau-t)dF_T(t)\right)+C_R}{\sum_{i=1}^{\infty}i\tau(F_T(i\tau)-F_T((i-1)\tau))}. \tag{27}$$

To minimize the average maintenance cost rate in long run, we can calculate the derivative of $AVC(\tau)$, as detailed in Appendix 1. By setting $AVC'(\tau)=0$, the optimal interval τ can be obtained for the periodic inspection policy.

4.2. Age-based replacement policy

Under this maintenance policy, the system is replaced at a specified age Ψ without any inspection. When the system fails before Ψ, there will be a period of delay time for the system until Ψ at a cost rate C_D, and the system will be correctively replaced with a cost C_R. Otherwise, the system will be preventively replaced with a cost C_P at Ψ. Both the preventive replacement and the corrective replacement restore the system to as-good-as new state.

In this case, the expected cost rate in long run is given by

$$AVC(\Psi)=\frac{C_R F_T(\Psi)+C_P(1-F_T(\Psi))+C_D\int_0^{\Psi}F_T(t)dt}{\Psi},\ \Psi>0. \tag{28}$$

When Ψ is very large, the cost rate will be large due to the high probability of failure and long delay time. On the other hand, when Ψ is very small, the cost rate will also be large due to the high frequency of preventive replacement. Therefore there exists an optimal Ψ to achieve the minimum expected cost rate. The derivation of Eq. (28) is given by

$$AVC'(\Psi)=\frac{(C_R f_T(\Psi)-C_P f_T(\Psi)+C_D F_T(\Psi))\Psi-(C_R F_T(\Psi)+C_P(1-F_T(\Psi))+C_D\int_0^{\Psi}F_T(t)dt)}{\Psi^2}. \tag{29}$$

By setting $AVC'(\Psi)=0$, the optimal Ψ can be obtained for the age-based maintenance policy.

5. Numerical example

In this section, the joint copula reliability model is constructed and the two maintenance policies are studied for a system subjected to two degradation processes and random shocks. The two degradation processes are governed by gamma processes with parameters $\alpha_1=0.2$, $\beta_1=2$, $\alpha_2=0.3$, $\beta_2=2$. The failure thresholds for the two degradation processes are $L_1=6$, $L_2=8$. The random shocks follow a NHPP process with $r=0.1$, $c=0.01$. The random shock damages to the two degradation processes follow $S_{1,k}\sim N(0.2,0.1^2)$ and $S_{2,k}\sim N(0.5,0.2^2)$, respectively. The cost parameters are assumed as follows: C_I=1 per inspection, C_P=180 per PM, C_R=200 per replacement, C_D=100 per unit time.

5.1 Copula function selection

According to the copula function properties, we can use the marginal reliability function in Eq. (9) to construct the joint reliability function with the underlying dependent relationship. With the given

Table 1. Simulated marginal reliability data with dependent relationship

Time	75	80	85	90	95	100	105	110	115
$R_1(t)$	0.9999	0.9999	0.9999	0.9970	0.9880	0.9590	0.9180	0.8700	0.8300
$R_2(t)$	0.9999	0.9999	0.9980	0.9890	0.9390	0.8780	0.7450	0.6130	0.4630
R(t)	0.9999	0.9999	0.9980	0.9860	0.9270	0.8370	0.6630	0.4830	0.2930
Time	120	125	130	135	140	145	150	155	160
$R_1(t)$	0.7680	0.7580	0.7410	0.7440	0.7650	0.7460	0.7500	0.7480	0.7350
$R_2(t)$	0.3960	0.3050	0.2960	0.2750	0.2500	0.2670	0.2590	0.2570	0.2760
R(t)	0.1640	0.0063	0.0037	0.0019	0.0015	0.0130	0.0090	0.0050	0.0110

Table. 2 Results of copula fitting for Type I reliability function

Copula type	Parameter (Θ)	LL	AIC	BIC	ARE
Gumbel	1.9902	15.2297	-28.4593	-27.5689	0.4242
Clayton	0.6376	**25.8857**	**-49.7713**	**-48.8809**	0.4994
Frank	2.9086	3.0513	-4.1027	-3.2123	0.4823
Gaussian	0.9816	20.8389	-39.6779	-38.7875	**0.4053**
Student's t	0.6103	22.2799	-42.5598	-41.6694	0.4945

Table 3. Results of copula fitting for Type II reliability function

Copula type	Parameter (Θ)	LL	AIC	BIC	ARE
Gumbel	1.3195	**25.6364**	**-49.2728**	**-48.3824**	0.4912
Clayton	0.94423	2.1686	-2.33719	-1.44682	0.4433
Frank	2.9086	3.05135	-4.1027	-3.2123	0.4823
Gaussian	0.9832	20.8389	-39.6779	-38.7875	**0.4053**
Student's t	0.3226	22.2799	-42.5598	-41.6694	0.4816

parameters, we can simulate the competing failure processes for the system to obtain the marginal reliability functions and the joint reliability function with dependent competing risks (see table 1). In this paper, Gumbel copula, Clayton copula, Frank copula, Gaussian copula and t-copula are employed to fit the joint reliability distribution. Based on the simulated marginal reliability data in table 1, we can use MLE to estimate the parameters of the copula functions.

Denote the joint reliability function in Eq. (15) by Type I reliability function, and the joint reliability function in Eq. (16) by Type II reliability function.

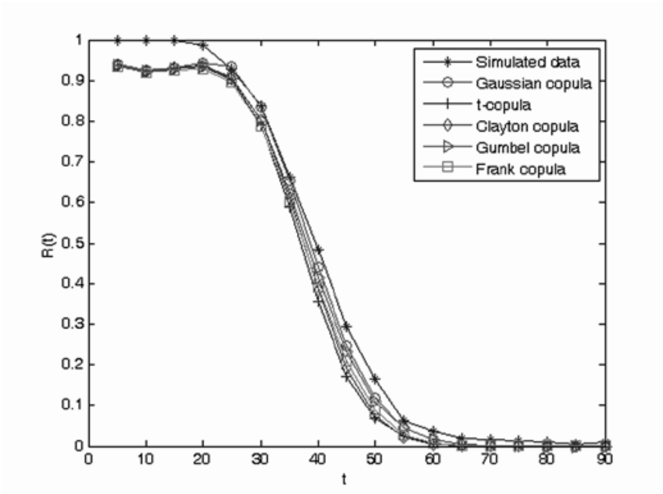
The results for Type I reliability function are given in table 2. The criteria, Log-likelihood (LL), Akaike information criterion (AIC) and Bayesian information criterion (BIC) are used to show the goodness of fit. AIC and BIC are two criterion functions with the difference that the BIC also takes account of the sample size. Besides, the average relative error (ARE) criterion is used to judge the relative error between the fitted reliability data and the simulated reliability data, and determine which copula function has the highest precision to estimate the system reliability. The ARE is computed as

ARE$=\frac{1}{N}\sum_{i=1}^{N}\left|\frac{R_{sim}(t_i)-R_{copula}(t_i)}{R_{sim}(t_i)}\right|$, where R_{sim} is the simulated reliability result, R_{copula} is the reliability computed with the copula function, t_i corresponds to the time in table 1 and N_1 =18 in this case.

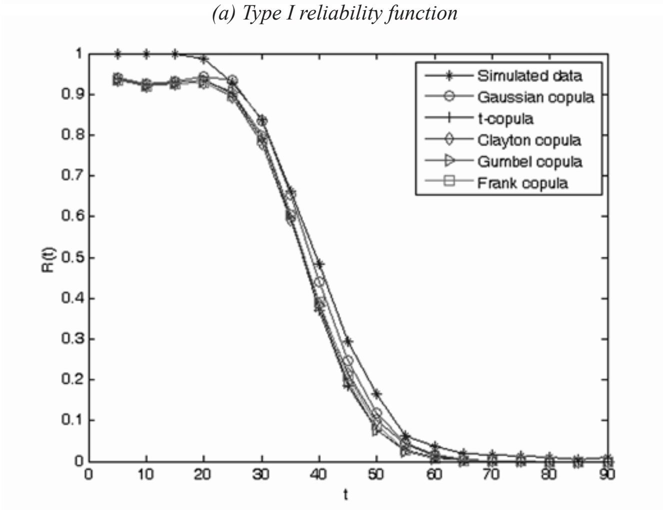
The results for Type II reliability function are given in table 3.

From the results in table 1 and 2, it can be seen that the Clayton Copula is the most suitable copula function for fitting Type I reliability function, the Gumbel Copula is the most suitable copula function for fitting Type II reliability function, but Gaussian copula has the highest precision for the system reliability estimation with Type I or Type II reliability function. The comparisons of the joint copula reliability functions are shown in Fig.1.

Through the comparison in Fig.1, we can see that the Gaussian copula is obviously better than other copula functions. Therefore, Gaussian copula is chosen to model the joint reliability of the sys-



(a) Type I reliability function



(b) Type II reliability function

Fig.1. Comparison between the copula reliability with the simulated system reliability data

tem with dependent competing risks and Type II reliability function is chosen as the system reliability function for simplicity.

Fig. 2 shows the comparison between the reliability computed by Gaussian copula reliability function in Eq. (16) and the independent reliability function in Eq. (12). It is shown that the proposed joint reliability function provides more precise results than the independent reliability function.

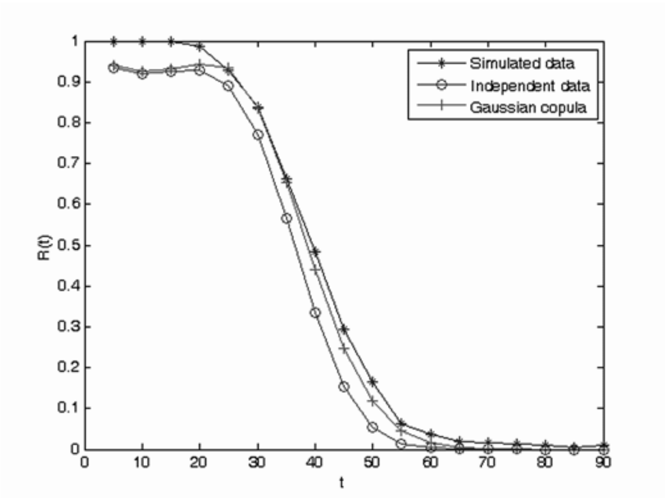


Fig.2. Gaussian copula reliability versus independent reliability

5.2. Maintenance optimization

Based on the Gaussian copula reliability function, we can use Eq. (27) to obtain the optimal inspection interval τ^*=0.2 with minimum $AVC(\tau^*)$ =15. Fig.3 illustrates the average maintenance cost rate in long run as a function of τ.

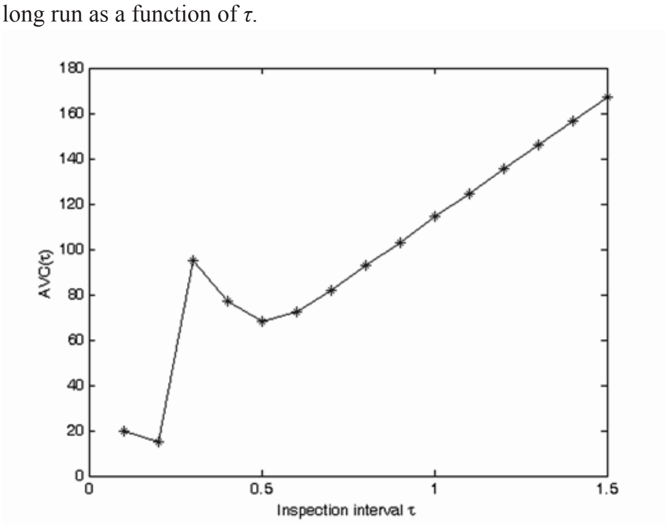


Fig.3. Average long-run maintenance cost rate versus inspection interval τ with C_I=1

For the age-based maintenance policy, we can use Eq. (29) to obtain the optimal replacement interval Ψ^*=28 with the minimum expected cost rate achieved as 16.17. Fig.4 depicts the expected cost rate in long run as a function of Ψ.

By comparing the optimal results of the two maintenance policies, it is found that the periodic inspection/replacement policy is more profitable than the age-based replacement policy (15<16.17). Actually, appropriate inspection plan can effectively reduce the maintenance cost when the inspection cost is not very high. However, when the inspection action costs too much, the periodic inspection/replacement policy will not show superiority over the age-based maintenance policy. Fig.5 shows the maintenance cost rate versus the inspection interval with C_I=20. The optimal cost rate $AVC(\tau^*)$ =60>16.17 is achieved at t^*=0.2. This proves that the inspection cost is an important factor for choosing the maintenance policy.

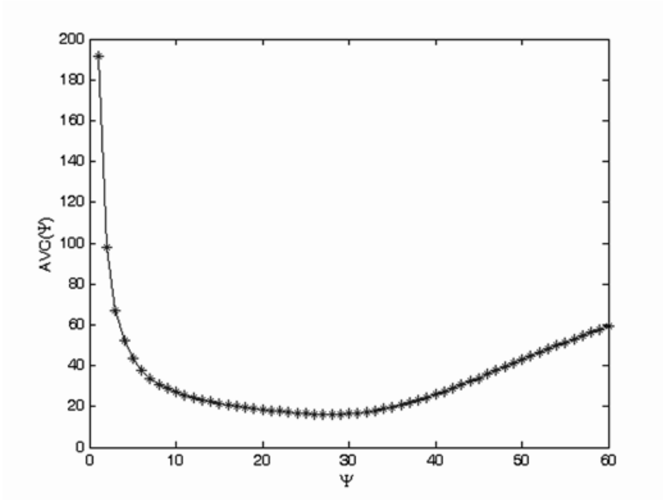


Fig.4. Evolution of expected cost rate versus

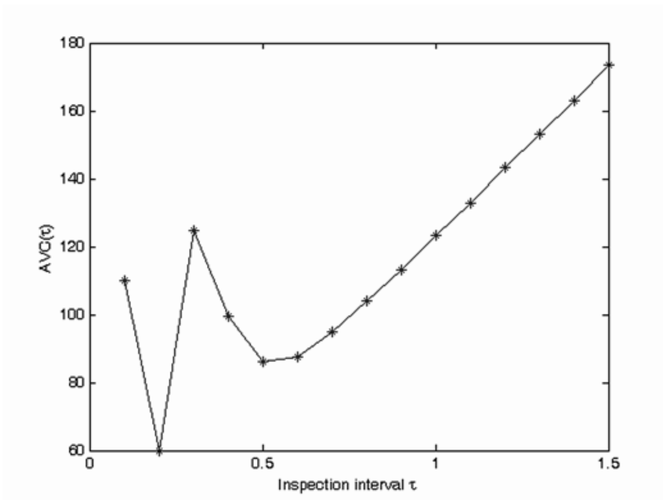


Fig.5. Average long-run maintenance cost rate versus inspection interval τ with C_I=20

6. Conclusions

In this paper, we developed a joint copula reliability model for dependent competing risks with two degradation processes and random shocks. The random shocks can cause additional shock damages to the two degradation processes. A two-stage estimation method is proposed to estimate the parameters of the copula function based on simulated data. Gaussian copula function is chosen to model the system reliability with multiple dependent competing risks judging by the evaluation criteria. Based on the copula reliability model, we studied two maintenance policies for a non-repairable system. Through comparison, we find that the periodic inspection/replacement policy is superior over the age-based maintenance policy when the inspection cost is low. But when the inspection cost is high, the age-based maintenance policy will be more profitable than the periodic inspection/replacement policy.

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Appendix 1

$$AVC'(\tau)=\frac{uv'-u'v}{u^2}$$

$$u=\sum_{i=1}^{\infty} i\tau\left(F_T(i\tau)-F_T((i-1)\tau)\right),$$

$$u'=\sum_{i=1}^{\infty}\left[i(F_T(i\tau)-F_T((i-1)\tau))+i^2\tau f_T(i\tau)-i(i-1)\tau f_T((i-1)\tau)\right],$$

$$v=C_I\sum_{i=1}^{\infty} i(F_T(i\tau)-F_T((i-1)\tau))+C_D\sum_{i=1}^{\infty}\left(\int_{(i-1)\tau}^{i\tau}(i\tau-t)dF_T(t)\right)+C_R,$$

$$v'=C_I\sum_{i=1}^{\infty}(i^2 f_T(i\tau)-i(i-1)f_T((i-1)\tau))+C_D\sum_{i=1}^{\infty}\left(i\int_{(i-1)\tau}^{i\tau} f_T(t)dt+(1-i)\tau f_T((i-1)\tau)\right).$$

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