1. Introduction

The availability, as a performance measure, is one of the most important indicators for characterizing a repairable system and its components. For repairable multi-state systems with various performance levels, the availability is more meaningful than reliability to measure the effectiveness of the system to satisfy consumer demand.

In the past few years, a variety of methods are available in the literature for analyzing the availability of repairable multi-state systems. The discrete time reliability models for general binary systems can be found in [4, 22, 34]. Eryilmaz [14], Guerry [16] and Sadek and Limnios [35], presented the discrete time reliability models for Markov multi-state systems. The discrete time reliability models for semi-Markov multi-state systems were investigated in [1, 4, 10]. However, most of the reported works mainly focus on the issues of on a continuous scale. In such situations, the lifetimes and repair times length of some systems (components) can be expressed in terms of the number of working and repairing periods (cycles), respectively. Thus, it is essential to construct discrete time reliability models for repairable multi-state systems.

The discrete time reliability has drawn continuous attention in both model analysis and problem solution. Braquemond and Gaudoin [5] presented a good overview of discrete probability distributions used in reliability for modeling discrete lifetimes of non-repairable systems. The discrete time reliability modeling for general binary systems can be found in [4, 22, 34]. Eryilmaz [14], Guerry [16] and Sadek and Limnios [35], presented the discrete time reliability models for Markov multi-state systems. The discrete time reliability models for semi-Markov multi-state systems were investigated in [1, 4, 10]. However, most of the reported works mainly focus on the issues of

Fuzzy Availability Assessment for Discrete Time Multi-State System under Minor Failures and Repairs by Using Fuzzy $L_z$-Transform

This paper studies assessment approach of dynamic fuzzy availability for a discrete time multi-state system under minor failures and repairs. Traditionally, it was assumed that the exact reliability data of a component/system with discrete time are given in reliability analysis. In practical engineering, it is difficult to obtain precise data to evaluate the characteristics of a component/system. To overcome the problem, fuzzy set theory is employed to deal with dynamic availability assessment for a discrete time multi-state system in this paper. A fuzzy discrete time Markov model with fuzzy transition probability matrix is proposed to analyze the fuzzy state probability of each component at any discrete time. The fuzzy $L_z$-transform of the discrete-state discrete-time fuzzy Markov chain is developed to extend the $L_z$-transform of the discrete-state continuous-time Markov model with crisp sets. Based on the $\alpha$-cut approach and the fuzzy $L_z$-transform, the dynamic fuzzy availability of the system is computed by using parametric programming technique. To illustrate the proposed method, a flow transmission system is analyzed as a numerical example.

Keywords: discrete time, Markov model, fuzzy $L_z$-transform, multi-state system, availability.
discrete time systems with the exact reliability data. As stated in Garg [15], the complicated system has the massive fuzzy uncertainty due to which it is difficult to get the exact probability of the events. Thus, it is of large practical value to investigate the availability assessment for discrete time repairable system with fuzzy uncertainty.

Fuzzy reliability theory, which is based on the fuzzy set theory introduced by Zadeh [39,40], is becoming a useful tool for dealing with the imprecision and uncertainty problems of reliability evaluation for many industrial systems. The basic concept and theory of the fuzzy reliability have been introduced and developed by several authors [7, 18, 27, 38]. More recently, fuzzy reliability research has focused on reliability/availability evaluation of fuzzy multi-state system according to various analysis methods. Ding and Lissianski [12] firstly provided the basic definition of the fuzzy multi-state system model, and then investigated the system reliability based on the proposed fuzzy universal generating function technique. The concepts of relevancy, coherency, equivalence, and performance evaluation algorithms for the fuzzy multi-state systems were given by Ding, Zuo, Lissianski and Tian [13]. Liu and Huang [33] proposed a fuzzy Markov model to establish dynamic state probabilities of fuzzy multi-state elements, and investigated a dynamic fuzzy reliability assessment method for fuzzy multi-state systems. Li, Chen, Yi and Tao [25] developed interval universal generating function to analyze the reliability of multi-state systems when the available data of components are insufficient. Bamrunsetthapong and Pongpulponsak [3] studied the fuzzy system reliability for a non-repairable multi-state series-parallel system by using fuzzy Bayesian inference based on prior interval probabilities. Hu, Yue and Tian [17] provided a special assessment approach for evaluating the fuzzy steady-state availability of a repairable multi-state series-parallel system based on fuzzy universal generating function and parametric programming technique.

However, all the reported works for fuzzy multi-state system reliability mainly focus on the issues of dynamic reliability/availability assessment for continuous time multi-state systems or steady-state availability assessment for repairable multi-state systems. The fuzzy reliability/availability assessment for a discrete-time multi-state system has been seldom discussed in the literature. Moreover, availability has a wider scope than reliability as it takes into account maintenance time analysis in addition to failure time analysis [8]. Therefore, the main objective of our work is to present an analytical technique of dynamic fuzzy availability assessment for a discrete time repairable multi-state system (DTRMSS) with fuzzy consumer demand. The technique called the fuzzy $L_c$-transform is based on the combination of fuzzy universal generating function technique and discrete time Markov process method. In the presented paper, the fuzzy $L_c$-transform for discrete-time discrete-time Markov process is developed to extend the $L_c$-transform for discrete-state continuous-time Markov process [28] with crisp sets. Minor failures and repairs of components [30] (that transitions can only occur between adjacent states) with fuzzy state transition probabilities are considered. Fuzzy discrete-time discrete-time Markov process model is proposed to perform fuzzy state probability analysis for each component in dynamic modes. The dynamic fuzzy state probability, dynamic fuzzy performance level and dynamic fuzzy availability of the system are evaluated by the proposed fuzzy $L_c$-transform method, and the $\alpha$-cut of dynamic fuzzy availability is computed according to parametric programming technique.

The rest of this paper is organized as follows. The discrete time fuzzy Markov model for a repairable multi-state component is presented in Section 2. Section 3 describes the definition of fuzzy $L_c$-transform. The fuzzy dynamic availability assessment method for the DTRMSS is given in Section 4. The analytical technique and assessment method are illustrated in Section 5 via a flow transmission system. Conclusions are given in Section 6.

2. Discrete time repairable multi-state component

2.1. Markov model for repairable multi-state component

Consider a discrete time repairable multi-state component with $m$ different possible states $i$ ($i = 1, 2, \ldots, m$), where 1 and $m$ represent perfect function and complete failure states, respectively. Assume that these states of the component correspond to different performance levels $x_{1}, x_{2}, \ldots, x_{m}$, where $x_{i}$ is the performance level associated with the state $i$. Let $x(k)$ denote the performance level of the component at the end of the $k$th time period (such as hour, day, month, etc.), $x(k)$ takes values from $\{x_{1}, x_{2}, \ldots, x_{m}\}$ with $x_{i}\geq 0$, $i = 1, 2, \ldots, m$, $k = 0, 1, \ldots$. Thus, the performance level $x(k)$ is a discrete-state discrete-time stochastic process.

For the repairable multi-state component, we assume that minor failures and repairs [30] are considered. Minor failures are failures causing component transition from state $i$ to the adjacent state $i-1$, and minor repairs are repairs causing component transition from state $i$ to the adjacent state $i+1$. In some components development process, the life of the components need to test. The geometric distribution has an important application in testing the life of the components. Sarhan, Guess and Usher [36] stated that the geometric distribution is a common discrete distribution used to model the lifetime of a device. Eryilmaz [14] investigated the mean residual life of discrete time multi-state systems based on the geometric distribution. In our work, it is assumed that the time between transitions from one state to another has geometric distribution with constant mean values $1/p_{i,i+1}$ and $1/p_{i,i-1}$. Let $T_{i}$ and $T_{i}$ denote the time between transitions from the state $i$ to the state $i+1$ and from the state $i$ to the state $i-1$, respectively. We have:

$$P\{T_{i} = k\} = q_{i,i+1}(1-q_{i,i+1})^{k-1},\quad i = 1, 2, \ldots, m-1,\ k = 1, 2, \ldots$$

and:

$$P\{T_{i} = k\} = p_{i,i-1}(1-p_{i,i-1})^{k-1},\quad i = 2, 3, \ldots, m,\ k = 1, 2, \ldots$$

It is obvious that the process $x(k)$ has the property of being memory-less. Furthermore, for the repairable multi-state component, its performance level $x(k)$ ($k = 0, 1, \ldots$) is a discrete-state discrete-time Markov chain with the following one-step transition probability matrix:

$$P = \begin{bmatrix}
P_{11} & P_{12} & 0 & \cdots & 0 & 0 & 0 \\
P_{21} & P_{22} & P_{23} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & P_{(m-1)(m-2)} & P_{(m-1)(m-1)} & P_{(m-1)m} \\
0 & 0 & 0 & \cdots & 0 & P_{m(m-1)} & P_{mm}
\end{bmatrix}$$

That is, the one-step transition probability $P_{ij}$ from component state $i$ to component state $j$ is determined by:

$$P_{ij} = P\{x(k+1) = j | x(k) = i\} = \begin{cases}
q_{i,j+1}, & j = i+1, \ i = 1, 2, \ldots, m-1 \\
p_{i-1 j}, & j = i-1, \ i = 2, 3, \ldots, m \\
1-q_{i,j+1}, & j = i+1 \\
p_{i-1 j-1}, & j = i = m \\
p_{i j-1} - q_{i,j+1}, & j = i = 2, 3, \ldots, m-1 \\
0, & \text{otherwise}
\end{cases}$$
where $0 \leq q_{i,j+1} \leq 1 \ (i=1,2,\ldots,m-1)$, $0 \leq p_{i,i-1} \leq 1 \ (i=2,3,\ldots,m)$ and $0 \leq q_{i,j+1} + p_{i,i-1} \leq 1 \ (i=2,3,\ldots,m-1)$.

Let $P_k$ denote the probability that the component is in state $i (i=1,2,\ldots,m)$ at time $k$, the state probabilities of the component at any discrete time can be calculated by the matrix equation:

$$[P_1(k), P_2(k), \ldots, P_m(k)] = [P_1(k-1), P_2(k-1), \ldots, P_m(k-1)] \cdot P \quad (2)$$

with the initial conditions:

$$P_1(0) = 1, \ P_2(0) = P_3(0) = \cdots = P_m(0) = 0 \quad (3)$$

By solving the matrix equation (2) under the initial condition (3), we can determine the following matrix equation:

$$[P_1(k), P_2(k), \ldots, P_m(k)] = [P_1(0), P_2(0), \ldots, P_m(0)] \cdot P^k \quad (4)$$

The matrix equation (4) can be written as:

$$P_k = P_0 \cdot P^k \quad (5)$$

where $P_k = \begin{bmatrix} P_1(k), P_2(k), \ldots, P_m(k) \end{bmatrix}$ is the row vector of the state probabilities of the component at time $k$ and $P_0 = \begin{bmatrix} P_1(0), P_2(0), \ldots, P_m(0) \end{bmatrix} = \begin{bmatrix} 1,0,0,\ldots,0 \end{bmatrix}$ is the row vector of the state probabilities of the component at time 0.

### 2.2. Fuzzy Markov model for repairable multi-state component

In this subsection, we define the fuzzy discrete time repairable multi-state component and propose a discrete time fuzzy Markov model to evaluate the dynamic fuzzy state probability for the discrete time repairable multi-state component.

#### 2.2.1. Definition and assumption

The fuzzy discrete time repairable multi-state component is defined as the component in which the different state performance levels, the corresponding state probabilities or one-step transition probabilities between each pair of adjacent states are represented as fuzzy values. The general assumptions of a fuzzy discrete time repairable multi-state component are given as follows:

1. (State index of the component) is a crisp value taking integer values only, and the state space is $\{1,2,\ldots,m\}$.
2. (State performance level $x(k)$ of the component at time $k$ $(k=0,1,\ldots)$ can be measured as fuzzy value. We substitute $\tilde{x}(k)$ for $x(k)$ in the subsection 2.1 to denote the fuzzy performance level $12, 17, 33$ of the component at time $k$, and $\tilde{x}_i$ for $x_i$ in the subsection 2.1 to denote the fuzzy performance level associated with the state $i$. $i=1,2,\ldots,m$. The fuzzy performance level $\tilde{x}(k)$ takes values from $[x_1, x_2, \ldots, x_m]$. Because the $x_i$ value is presented as fuzzy number in the model, we have $x_i \in \tilde{c}_i$ with $x_i \geq 0$, where $\tilde{c}_i$ is an $\alpha$-cut of the fuzzy number $x_i$ for $0 \leq \alpha \leq 1$.

(3) One-step transition probability of the component from one state to another state (that transitions can only occur between adjacent states) can be measured as fuzzy value. We substitute $\tilde{P}_{ij}$ for $P_{ij}$ in Eq. (1) to denote the fuzzy one-step transition probability $2, 23, 26$ from the component state $i$ to the component state $j \ (i=1,2,\ldots,m-1)$ and $p_{i,i-1} \ (i=2,3,\ldots,m)$ values are fuzzy values and we substitute $\tilde{q}_{i,j+1}$ for $q_{i,j+1}$ and $\tilde{p}_{i,i-1}$ for $p_{i,i-1}$ in Eq. (1). $P_k$ can be determined by the fuzzy values $\tilde{q}_{i,j+1}$ and $\tilde{p}_{i,i-1}$ based on Eq. (1). Because the $q_{i,j+1}$ and $p_{i,i-1}$ values are presented as fuzzy numbers in the model, we put the following restrictions on the $\tilde{q}_{i,j+1}$ and $\tilde{p}_{i,i-1}$ values:

$$q_{i,j+1} \in \tilde{q}_{i,j+1} \quad \text{and} \quad p_{i,i-1} \in \tilde{p}_{i,i-1} \quad \text{with} \quad 0 \leq q_{i,j+1} \leq 1, \ 0 \leq p_{i,i-1} \leq 1 \quad \text{and} \quad 0 \leq q_{i,j+1} + p_{i,i-1} \leq 1$$

#### 2.2.2. Fuzzy Markov model

According to definition and assumptions of the fuzzy discrete time repairable multi-state component, the fuzzy performance level $\tilde{x}(k) \ (k=0,1,\ldots)$ forms a discrete-state discrete-time fuzzy Markov chain $2, 23, 26$ with the following fuzzy one-step transition probability matrix:

$$\tilde{P} = \begin{bmatrix} 1-q_{1,2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ q_{1,2} & 1-p_{2,1} & \cdots & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & \ddots & \ddots & 0 \end{bmatrix}$$

where the fuzzy uncertainty is on the state transition probabilities of the component from $i$ to the state $i+1$ and from the state $i$ to the state $i-1$, but not on the fact that every row must add to 1. The fuzzy values $\tilde{q}_{i,j+1}$ and $\tilde{p}_{i,i-1}$ are restricted by $\tilde{q}_{i,j+1} \in \tilde{q}_{i,j+1}$ and $\tilde{p}_{i,i-1} \in \tilde{p}_{i,i-1}$ with $0 \leq q_{i,j+1} + p_{i,i-1} \leq 1$.

With the fuzzy state transition probabilities, the state probability of the component in the state $i$ at time $k$ must also be a fuzzy value denoted as $\tilde{P}_k(i)$. The dynamic fuzzy state probability $\tilde{P}_k(i)$ can be determined by:

$$[\tilde{P}_1(k), \tilde{P}_2(k), \ldots, \tilde{P}_m(k)] = [\tilde{P}_1(0), \tilde{P}_2(0), \ldots, \tilde{P}_m(0)] \cdot P^k \quad (6)$$

The matrix equation (6) can be written as:

$$\tilde{P}_k = \tilde{P}_0 \cdot P^k \quad (7)$$

where $\tilde{P}_k = \begin{bmatrix} \tilde{P}_1(k), \tilde{P}_2(k), \ldots, \tilde{P}_m(k) \end{bmatrix}$ and $\tilde{P}_0 = \begin{bmatrix} 0,0,\ldots,0 \end{bmatrix}$. By solving the matrix equations (6) or (7), the dynamic fuzzy state probability $\tilde{P}_k(k)$ at time $k$ can be given as function of fuzzy variables $\tilde{q}_{i,j+1}(i=1,2,\ldots,m-1)$ and $\tilde{p}_{i,i-1}(i=2,3,\ldots,m)$, and then $\tilde{P}_k(k)$ can be written as $\tilde{P}_k(k) = P_k \cdot \tilde{q}_{i,j+1} \cdot \tilde{p}_{i,i-1} \cdot \ldots \cdot \tilde{P}_m(k)$.

If the fuzzy state transition probabilities are represented by fuzzy vectors $\tilde{q} = \{q_{1,2}, q_{2,3}, \ldots, q_{m-1,m} \}$ and $\tilde{p} = \{p_{1,1}, p_{2,2}, \ldots, p_{m,m} \}$, we have $\tilde{P}_k(k) = \tilde{P}_1(k, \tilde{q}, \tilde{p})$. 

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**EKSPLAOTACJA i NIEZAWODNOSC – MAINTENANCE AND RELIABILITY Vol.19, No. 2, 2017**

181
Let \( \eta_{q_{i+1}} \) and \( \eta_{p_{i-1}} \) denote the membership functions of \( q_{i+1} \) and \( p_{i-1} \), respectively. The \( \alpha \)-cuts of \( q_{i+1} \) and \( p_{i-1} \) can be determined as crisp intervals:

\[
\begin{align*}
(\hat{q}_{i+1})_\alpha &= \left[ \min_{q_{i+1}<q_i} q_{i+1}, \max_{q_{i+1}>q_i} q_{i+1} \right] \\
&= \left[ q_{i+1}^\alpha, q_{i+1}^{1-\alpha} \right], \\
(\hat{p}_{i-1})_\alpha &= \left[ \min_{p_{i-1}<p_i} p_{i-1}, \max_{p_{i-1}>p_i} p_{i-1} \right] \\
&= \left[ p_{i-1}^\alpha, p_{i-1}^{1-\alpha} \right],
\end{align*}
\]

(8)

(9)
where \( Q_{i+1} \) and \( P_{i-1} \) are the crisp universal sets of the state transition probabilities for the component from the state \( i \) to the state \( i+1 \) and from the state \( i \) to the state \( i-1 \), respectively. According to parametric programming technique [20, 33], the lower bound \( L_z(\hat{P}_i) \) and upper bound \( U_z(\hat{P}_i) \) of the \( \alpha \)-cut of \( \hat{P}_i(k, q, p) \) can be computed as:

\[
\begin{align*}
L_z(\hat{P}_i) &= \min_{\eta=0} P(\hat{x}_i, q, p), \\
&= \min_{q_i \leq q \leq q_i^\alpha, p \leq p \leq p_i^{1-\alpha}} P_q(k, q) P_p(k, p), \\
&= \left( \hat{P}_i \right)_\alpha^L, \quad (k = 1, 2, \ldots, 0 \leq \alpha \leq 1), \\
&\text{s.t.} \quad (q_i, q_i^\alpha) \leq (q, q_i^\alpha), \quad i = 1, 2, \ldots, m-1 \quad (10)
\end{align*}
\]

and:

\[
\begin{align*}
U_z(\hat{P}_i) &= \max_{\eta=0} P(\hat{x}_i, q, p), \\
&= \max_{q_i \leq q \leq q_i^\alpha, p \leq p \leq p_i^{1-\alpha}} P_q(k, q) P_p(k, p), \\
&= \left( \hat{P}_i \right)_\alpha^U, \quad (k = 1, 2, \ldots, 0 \leq \alpha \leq 1), \\
&\text{s.t.} \quad (q_i, q_i^\alpha) \leq (q, q_i^\alpha), \quad i = 1, 2, \ldots, m-1 \quad (11)
\end{align*}
\]

where \( q = \{q_1, q_2, \ldots, q_{m-1, m}\} \) and \( P = \{P_1, P_2, \ldots, P_{m-1}\} \) are crisp state transition probability vectors. Then \( \alpha \)-cut of \( \hat{P}_i(k) \) can be denoted as \( \hat{P}_i(k) \alpha = \left( \hat{P}_i(k) \right)_\alpha^L, \quad \left( \hat{P}_i(k) \right)_\alpha^U \).

3. Fuzzy \( L_z \)-transform

Consider the discrete-state discrete-time fuzzy Markov chain \( \hat{x}(k)(k = 0, 1, \ldots) \) in subsection 2.2, the fuzzy Markov chain can be completely determined by set of possible fuzzy performance levels \( \hat{x} = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m\} \), fuzzy transition probability matrix \( \hat{P} \) and the initial state probability distribution \( \hat{P}_0 \). The discrete-state discrete-time fuzzy Markov chain can be denoted by using triplet:

\[
\hat{x}(k) = (\hat{x}, \hat{P}, \hat{P}_0).
\]

Based on the fuzzy universal generating function [12, 33] and the \( L_z \)-transform of a discrete-state continuous-time Markov process [28, 29, 31], fuzzy \( L_z \)-transform of a discrete-state discrete-time fuzzy Markov chain \( \hat{x}(k) \) is defined as:

\[
\hat{L}_z(\hat{x}(k)) = \sum_{i=0}^{\infty} \hat{P}_i(k) \cdot z^i
\]

where \( \hat{P}_i(k) \) is the fuzzy state probability that the fuzzy Markov chain is in the state \( i \) at time \( k(k = 0, 1, \ldots) \) for any given initial state probability distribution \( \hat{P}_0 \), and \( z \) is a complex variable in the general case. Under given initial state probability distribution \( \hat{P}_0 \), the discrete-state discrete-time fuzzy Markov chain has one and only one fuzzy \( L_z \)-transform.

For example, consider a simple component which has only two different fuzzy performance levels \( \hat{x}_1 = \hat{X}_1 \) and \( \hat{x}_2 = 0 \). It means that \( \hat{x}_1 \) is the performance level associated with nominal working state and \( \hat{x}_2 \) is the performance level associated with complete failure state. The working time and the repair time of the component have geometric distributions with fuzzy mean values \( 1/\hat{q} \) and \( 1/\hat{p} \). The fuzzy Markov chain \( \hat{x}(k) = \{\hat{x}_1, \hat{x}_2\} \) is the fuzzy state probability that the fuzzy Markov chain is in the state \( k(k = 0, 1, \ldots) \) for the considered example is denoted by \( \hat{x}(k) = (\hat{x}, \hat{P}, \hat{P}_0) \), where \( \hat{x}, \hat{P} \) and \( \hat{P}_0 \) are given respectively as:

- Set of the possible fuzzy performance levels \( \hat{x} = \{\hat{x}_1, \hat{x}_2\} = \{X_1, 0\} \};
- Fuzzy transition probability matrix \( \hat{P} = \begin{bmatrix} 1-\hat{q} & \hat{q} \\ \hat{p} & 1-\hat{p} \end{bmatrix} \);
- Initial state probability distribution \( \hat{P}_0 = [1, 0] \).

The fuzzy dynamic state probabilities of the fuzzy Markov chain \( \hat{x}(k) \) at time \( k(k = 0, 1, \ldots) \) can be calculated by:

\[
\begin{bmatrix} \hat{P}_1(k), \hat{P}_2(k) \end{bmatrix} = [1, 0] \begin{bmatrix} 1-\hat{q} & \hat{q} \\ \hat{p} & 1-\hat{p} \end{bmatrix}^k
\]

Solving (14), we can obtain:

\[
\hat{P}_1(k) = \frac{\hat{p} + \hat{q}(1-\hat{q}-\hat{p})^k}{\hat{q} + \hat{p}}, \quad \hat{P}_2(k) = \frac{-\hat{q}(1-\hat{q}-\hat{p})^k}{\hat{q} + \hat{p}}
\]

For the component with a number of fuzzy performance levels, the forms of closed-form solutions for the dynamic fuzzy state probabilities \( \hat{P}_i(k) (i = 1, 2, \ldots, m) \) are very complicated. A numerical technique can be used to obtain these solutions.

The fuzzy \( L_z \)-transform of the fuzzy Markov chain for the binary component can be obtained as follows:
\[ \bar{L}_Z \left[ \bar{x}(k) \right] = \tilde{u}(z, k, \mathbf{P}_0) = \frac{2}{n} \sum_{i=1}^{M} \bar{P}_i(k) \cdot z^{\bar{\eta}_i} \]

\[ = \frac{\bar{p} + q (1 - \bar{q} - \bar{p})^k}{q + p} \cdot z^{\bar{\eta} + \frac{q - \bar{q} (1 - \bar{q} - \bar{p})^k}{q + p}} \cdot z^0 \]

Assume that the \( \alpha \)-cuts of \( \tilde{q} \) and \( \tilde{p} \) are \( [q^U, q^L] \) and \( [p^U, p^L] \) respectively, we can obtain the \( \alpha \)-cuts of the fuzzy dynamic state probabilities \( \bar{P}_1(k) \) and \( \bar{P}_2(k) \) for the fuzzy Markov chain according to (10) and (11).

4. Fuzzy dynamic availability assessment for DTRMSS

The fuzzy dynamic availability assessment method for the DTRMSS under minor failures and repairs, which is based on using the proposed fuzzy \( L_z \)-transform. It is assumed that the behavior of any component \( l (l \in [1, 2, \ldots, n]) \) in the DTRMSS with \( n \) components can be characterized by the discrete-state discrete-time fuzzy Markov chain \( \bar{G}_l(k) \), which has \( m_l \) different states that correspond to different fuzzy performance levels represented by the ordered fuzzy set \( \bar{g}_l = [\bar{g}_{1l}, \bar{g}_{2l}, \ldots, \bar{g}_{ml}] \), where \( \bar{g}_{ij} \) is the fuzzy performance level of component \( l \) in its state \( i_j (l = 1, 2, \ldots, m_l) \).

Let \( \bar{P}_l = [\bar{P}_1(k), \bar{P}_2(k), \ldots, \bar{P}_{m_l}(k)] \) denote the fuzzy dynamic state probabilities associated with different states for the component \( l \) at time \( k \), that is:

\[ \bar{P}_{ij}(k) = \text{Pr}(\bar{G}_l(k) = \bar{g}_{ij}) \text{, } i_j = 1, 2, \ldots, m_l \text{, } l \in [1, 2, \ldots, n] \quad (15) \]

The fuzzy dynamic state probabilities \( \bar{P}_{ij}(k) \) for each of \( m_l \) states can be obtained by writing and solving a corresponding fuzzy matrix equation (7) with the given initial conditions \( \mathbf{P}_0 \).

Based on the fuzzy \( L_z \)-transform method, each discrete-state discrete-time fuzzy Markov chain \( \bar{G}_l(k) \) associated with the fuzzy output Markov process of the component \( l \) \( (l \in [1, 2, \ldots, n]) \) should be expressed as:

\[ \bar{L}_Z (G_l(k)) = \tilde{u}(z, k, \mathbf{P}_0) = \sum_{i=1}^{m_l} \bar{P}_{ij}(k) \cdot z^{\bar{\eta}_{ij}}, l = 1, 2, \ldots, n \quad (16) \]

For the entire DTRMSS, its states are separated through its fuzzy performance levels, which are unambiguously determined by the fuzzy performance levels of components and its structure. Assume that the DTRMSS has \( M \) different states and \( \bar{g}_j \) is the fuzzy performance level of the system in state \( j = 1, 2, \ldots, M \). Let \( G(i) \) denote the fuzzy performance level of the DTRMSS at time \( k \), then \( G(k) \) is a fuzzy stochastic process that takes fuzzy values from the set \( \mathbf{G} = [\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_M] \). Based on the fuzzy performance stochastic processes \( \bar{G}_l(k) \ (l \in [1, 2, \ldots, n]) \) of all components at time \( k \) and the system structure, the fuzzy stochastic process \( G(k) \) can be given by:

\[ \tilde{G}(k) = \phi(\bar{G}_1(k), \bar{G}_2(k), \ldots, \bar{G}_n(k)) \quad (17) \]

where \( \phi(*) \) is the system structure function. Let \( \tilde{P}_j(k) \) denote the fuzzy dynamic state probability of the DTRMSS in state \( j \) at time \( k \), then the fuzzy dynamic state probabilities associated with different states for the system at time \( k \) can be denoted by:

\[ \mathbf{P}(k) = [\tilde{P}_1(k), \tilde{P}_2(k), \ldots, \tilde{P}_M(k)] \quad (18) \]

Based on the property of \( L_z \)-transform \( [28] \) and the general fuzzy composition operator \( \Omega \), we can find the fuzzy \( L_z \)-transform of the discrete-state discrete-time fuzzy stochastic process \( \bar{G}(k) \), which is a fuzzy single-valued function of \( n \) independent discrete-state discrete-time fuzzy Markov chains \( \bar{G}_l(k) \ (l \in [1, 2, \ldots, n]) \).

Applying the fuzzy composition operator \( \Omega \) to all individual fuzzy \( L_z \)-transforms \( \tilde{L}_Z (G_l(k)) \ (l \in [1, 2, \ldots, n]) \) over any discrete time \( k \) \( (k = 0, 1, \ldots) \), we have:

\[ \tilde{L}_Z (G(k)) = \Omega \left[ \tilde{L}_Z (G_1(k)), \tilde{L}_Z (G_2(k)), \ldots, \tilde{L}_Z (G_n(k)) \right] \quad (19) \]

According to (16) and (19), the fuzzy \( L_z \)-transform of the discrete-state discrete-time fuzzy stochastic process \( G(k) \) can be written as:

\[ \tilde{L}_Z (G(k)) = \Omega \left[ \tilde{u}(z, k, \mathbf{P}_10), \tilde{u}(z, k, \mathbf{P}_20), \ldots, \tilde{u}(z, k, \mathbf{P}_n0) \right] \]

\[ = \Omega \left[ \sum_{i=1}^{m_1} \bar{P}_{i1}(k) \cdot z^{\bar{\eta}_{i1}}, \sum_{i=1}^{m_2} \bar{P}_{i2}(k) \cdot z^{\bar{\eta}_{i2}}, \ldots, \sum_{i=1}^{m_n} \bar{P}_{in}(k) \cdot z^{\bar{\eta}_{in}} \right] \]

\[ = \Omega \left[ \frac{m_1}{M} \sum_{i=1}^{m_1} \sum_{l=1}^{m_1} \bar{P}_{il}(k) \cdot z^{\bar{\eta}_{il}} \right] \]

\[ \leq \sum_{j=1}^{M} \bar{P}_j(k) \cdot z^{\bar{\eta}_j} \quad (20) \]

where \( \bar{P}_j(k) = \sum_{l=1}^{M} \bar{P}_{il}(k) \) and \( \bar{g}_j = \phi(\bar{g}_{1l}, \bar{g}_{2l}, \ldots, \bar{g}_{ml}) \). The lower and upper bounds of the \( \alpha \)-cuts of the fuzzy dynamic state probability \( \bar{P}_j(k) \) and the fuzzy performance level \( \bar{g}_j \) of the DTRMSS in state \( j \) at time \( k \) can be determined according to parametric programming technique.

The dynamic fuzzy availability of the entire DTRMSS for the fuzzy consumer demand \( \tilde{\omega} \) at time \( k \) \( (k = 0, 1, \ldots) \) is defined as:

\[ \tilde{A}(\tilde{\omega}, k) = \frac{M}{\sum_{j=1}^{M} \bar{P}_j(k) \cdot P(\bar{g}_j \geq \tilde{\omega})} \quad (21) \]

where \( P(\bar{g}_j \geq \tilde{\omega}) \) denotes the possibility of \( \bar{g}_j \geq \tilde{\omega} \). Because the \( \bar{g}_j \) and \( \tilde{\omega} \) values are fuzzy numbers, we give a method for
\( P(\tilde{g}_j \geq \tilde{\omega}) \) based on method of interval number ranking in accordance with possibility [11] and the \( \alpha \)-cuts of \( \tilde{g}_j \) and \( \tilde{\omega} \). We define the possibility of \( \tilde{g}_j \geq \tilde{\omega} \) as:

\[
p(\tilde{g}_j \geq \tilde{\omega}) = \max \left[ 0, (g_{ja}^U - (g_{ja}^L + (\omega)_{ja}^L - (\omega)_{ja}^U - \max [0, (\omega)_{ja}^L - (x_{ja}^L)]) \right]
\]

(22)

where \( (g_{ja}^L) \) and \( (g_{ja}^U) \) are the lower bound and upper bound of the fuzzy performance level \( \alpha \)-cut of \( \tilde{g}_j \), and \( (\omega)_{ja}^L \) and \( (\omega)_{ja}^U \) are the lower bound and upper bound of the \( \alpha \)-cut of the fuzzy consumer demand \( \tilde{\omega} \), respectively.

In order to find the \( \alpha \)-cut of the dynamic fuzzy availability (21), the following procedures can be adopted.

1. For each repairable multi-state component \( l \) \( (i \in [1,2,\ldots,n]) \) under minor failures and repairs, the \( \alpha \)-cuts \( (0 \leq \alpha \leq 1) \) of the fuzzy state transition probabilities \( \tilde{q}_{lj,i+1} \) \( (i_j = 1,2,\ldots,m_l - 1) \) from the state \( i_j \) to the state \( i_j + 1 \), \( \tilde{p}_{lj,i-1} \) \( (i_j = 2,3,\ldots,m_l) \) from the state \( i_j \) to the state \( i_j - 1 \), and the fuzzy performance level \( \tilde{g}_{lj} \) in state \( i_j \) \( (i_j = 1,2,\ldots,m_l) \) can be determined according to individual membership functions as follows:

\[
(\tilde{q}_{lj,i+1})_{ja}^L = \left[ (q_{lj,i+1})_{ja}^L + (q_{lj,i+1})_{ja}^U \right]
\]

(23)

\[
(\tilde{p}_{lj,i-1})_{ja}^L = \left[ (p_{lj,i-1})_{ja}^L + (p_{lj,i-1})_{ja}^U \right]
\]

(24)

\[
(\tilde{g}_{lj})_{ja}^U = \left[ (g_{lj})_{ja}^L + (g_{lj})_{ja}^U \right]
\]

(25)

2. Based on the proposed fuzzy Markov model in the subsection 2.2.2, the fuzzy dynamic state probabilities \( \tilde{P}_{lj} (k) \) \( (i_j = 1,2,\ldots,m_l, l \in [1,2,\ldots,n]) \) for each component at time \( k \) can be determined. According to the general fuzzy composition operator \( \Omega_k \) [12] and the individual fuzzy \( L_z \)-transforms of all components at time \( k \) , the fuzzy \( L_z \)-transform \( \tilde{L}Z \left[ (G(k))^L \right] \) of the entire DTRMSS can be obtained.

3. According to parametric programming technique, the lower bound \( (P_j(k))^L_{ja} \) and upper bound \( (P_j(k))^U_{ja} \) of the \( \alpha \)-cut of the fuzzy dynamic state probability \( \tilde{P}_j (k) \) of the DTRMSS in state \( j \) at time \( k \) can be computed as:

\[
(P_j(k))^L_{ja} = \min \left[ P_j(k) - \sum_{i=1}^{n} P_{lj} (k) \right] \quad (k = 1,2,\ldots,0 \leq \alpha \leq 1)
\]

s.t. \( (q_{lj,i+1})_{ja}^L \leq (q_{lj,i+1})_{ja}^U \), \( i_j = 1,2,\ldots,m_l - 1 \)

\[
(P_j(k))^U_{ja} = \sum_{i=1}^{n} P_{lj} (k) \quad (k = 1,2,\ldots,0 \leq \alpha \leq 1)
\]

s.t. \( (q_{lj,i+1})_{ja}^L \leq (q_{lj,i+1})_{ja}^U \), \( i_j = 1,2,\ldots,m_l - 1 \)

(26)

and:

\[
(P_j(k))^L_{ja} = \max \left[ P_j(k) - \sum_{i=1}^{n} P_{lj} (k) \right] \quad (k = 1,2,\ldots,0 \leq \alpha \leq 1)
\]

s.t. \( (q_{lj,i+1})_{ja}^L \leq (q_{lj,i+1})_{ja}^U \), \( i_j = 1,2,\ldots,m_l - 1 \)

\[
(P_j(k))^U_{ja} = \sum_{i=1}^{n} P_{lj} (k) \quad (k = 1,2,\ldots,0 \leq \alpha \leq 1)
\]

s.t. \( (q_{lj,i+1})_{ja}^L \leq (q_{lj,i+1})_{ja}^U \), \( i_j = 1,2,\ldots,m_l - 1 \)

(27)

where \( P_{lj} (k) \) is the function of \( q_{lj,i+1} \) and \( p_{lj,i-1} \).

The lower bound \( (g_{ja}^L) \) and upper bound \( (g_{ja}^U) \) of the \( \alpha \)-cut of the fuzzy performance level \( \tilde{g}_j \) of the DTRMSS in state \( j \) can be computed as:

\[
(g_{ja}^L) = \min g_j = \phi (g_{lj}, g_{2j}, \ldots, g_{nj}) \quad (0 \leq \alpha \leq 1)
\]

s.t. \( (g_{lj})_{ja}^L \leq (g_{lj})_{ja}^U \), \( i_j = 1,2,\ldots,m_l \)

(28)

and:

\[
(g_{ja}^U) = \max g_j = \phi (g_{lj}, g_{2j}, \ldots, g_{nj}) \quad (0 \leq \alpha \leq 1)
\]

s.t. \( (g_{lj})_{ja}^L \leq (g_{lj})_{ja}^U \), \( i_j = 1,2,\ldots,m_l \)

(29)

(4) Based on the defined possibility of \( \tilde{g}_j \geq \tilde{\omega} \) in Eq. (22) and the defined availability \( \tilde{A}(\alpha,k) \) in Eq. (21), the lower bound \( \left( A(\alpha,k) \right)^L_{ja} \) and upper bound \( \left( A(\alpha,k) \right)^U_{ja} \) of the \( \alpha \)-cut of the dynamic system fuzzy availability \( \tilde{A}(\alpha,k) \) for the fuzzy consumer demand \( \tilde{\omega} \) at time \( k \) can be computed as:
and when becomes (perfect performance), a partial operation state (corresponding to the fuzzy performance level ), partial operation state (corresponding to the fuzzy performance level ), and complete failure state (corresponding to the performance level , ). The fuzzy state transition probabilities of the two components are

\[
q_1 = q = (0.04, 0.05, 0.06, 0.07), \quad q_2 = q_3 = (0.06, 0.07, 0.08, 0.09),
\]

\[
p = (0.4, 0.5, 0.6, 0.7), \quad \bar{p} = (0.5, 0.6, 0.7, 0.8).
\]

The CP subsystem has only one component with four states: two perfect operation states (corresponding to the fuzzy performance level ), partial operation state (corresponding to the fuzzy performance level ), and complete failure state (corresponding to the performance level ). The fuzzy state transition probabilities of the component are

\[
\begin{align*}
q &= (0.09, 0.1, 0.11, 0.12), \\
q_2 &= (0.05, 0.06, 0.07, 0.08), \\
\bar{q} &= (0.2, 0.3, 0.4, 0.5), \quad \bar{p} = (0.5, 0.6, 0.7, 0.8).
\end{align*}
\]

The HSS subsystem has two identical parallel components with three different states: perfect operation state (corresponding to the fuzzy performance level ), partial operation state (corresponding to the fuzzy performance level ), and complete failure state (corresponding to the performance level , ). The fuzzy state transition probabilities of the two components are

\[
q = (14, 16, 18, 20), \quad q_1 = q_2 = (7, 8, 9, 10), \quad \bar{q} = (8, 10, 12, 14).
\]

Based on different fuzzy consumer demand levels and , Tables 1-2 presents the -cuts of the dynamic fuzzy availability for the system with different fuzzy consumer demand levels: 

\[
\tilde{\omega} = (12, 15, 18, 21) \quad \text{and} \quad \tilde{\omega} = (8, 10, 12, 14).
\]

The HSS subsystem has two identical parallel components with three different states: perfect operation state (corresponding to the fuzzy performance level ), partial operation state (corresponding to the fuzzy performance level ), and complete failure state (corresponding to the performance level , ). The fuzzy state transition probabilities of the two components are

\[
q = (14, 16, 18, 20), \quad q_1 = q_2 = (7, 8, 9, 10), \quad \bar{q} = (8, 10, 12, 14).
\]

Assume that \( \tilde{a} = (a_1, a_2, a_3, a_4) \) and \( \tilde{b} = (b_1, b_2, b_3, b_4) \) are two trapezoidal fuzzy numbers, then:

\[
\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)
\]

\[
\min\{\tilde{a}, \tilde{b}\} = \left(\min\{a_1, b_1\}, \min\{a_2, b_2\}, \min\{a_3, b_3\}, \min\{a_4, b_4\}\right)
\]

The LSS subsystem consists of two different components connected in parallel, and their respective performance is allowed in only one of two principal states: perfect performance level (state 1) and zero performance level (state 2). The fuzzy state transition probabilities of each component from state 1 to state 2 are

\[
q_{LSS} = (0.020, 0.025, 0.030, 0.035) \quad \text{and} \quad q_{LSS} = (0.05, 0.06, 0.07, 0.08),
\]

respectively. The fuzzy state transition probabilities of each component from state 2 to state 1 are

\[
q_{LSS} = (0.30, 0.35, 0.40, 0.45) \quad \text{and} \quad q_{LSS} = (0.40, 0.50, 0.60, 0.70),
\]

respectively. The fuzzy performance levels of the first component are

\[
\tilde{g}_{11} = (4, 5, 6, 7) \quad (\text{perfect performance}) \quad \text{and} \quad \tilde{g}_{12} = 0 \quad (\text{zero performance}).
\]

The fuzzy performance levels of the second component are

\[
\tilde{g}_{21} = (8, 10, 12, 14) \quad (\text{perfect performance}) \quad \text{and} \quad \tilde{g}_{22} = 0 \quad (\text{zero performance}).
\]
6. Conclusions

The dynamic fuzzy availability assessment technique for a DTRMSS under minor failures and repairs is investigated in the paper. The technique is based on the Fuzzy $L_2$-transform of the discrete-state discrete-time Markov process, the $\alpha$-cut approach and the parametric programming algorithm. A discrete-state discrete-time fuzzy Markov model is proposed to analyze the dynamic fuzzy state probability for each multi-state component at any discrete time. The analytical procedure is provided to calculate the $\alpha$-cut of the dynamic fuzzy availability of the system. A flow transmission system with five components is considered, and it illustrates the performance of the proposed technique. This technique is suitable and effective for the dynamic fuzzy availability assessment of the DTRMSS with fuzzy uncertainty. In future, we will concern the fuzzy $L_2$-transform of the discrete-state discrete-time Markov process and its application to the fuzzy multi-state system reliability, and the dynamic fuzzy availability optimization design for the DTRMSS.

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**Fig. 1. Structure and component state-transition diagrams of the repairable multi-state flow transmission system**

**Fig. 2. The membership functions of the fuzzy availability**

(a) $k=2$

(b) $k=4$

(c) $k=8$

(d) $k=12$

**Fig. 2. The membership functions of the fuzzy availability** $\tilde{\alpha}_{1,k}$ **at** $k=2,4,8,12$
Fig. 3. The membership functions of the fuzzy availability $\tilde{A}(\omega_{1}, k)$ at $k=2,4,8,12$

Fig. 4. The fuzzy availability $\tilde{A}(\omega_{1}, k)$ with $\alpha=0.0, 0.5, 1.0$

Fig. 5. The fuzzy availability $\tilde{A}(\omega_{2}, k)$ with $\alpha=0.0, 0.5, 1.0$
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