The proper organized and reliable logistic support affects the execution of operational tasks. When the logistic activity is narrowed down to the supply activity, we can say that the basic elements are focused on providing the necessary supplies and services on the right time for the right money [23]. In the case of maintaining the operational processes of technical system, the supply stream consists of five elements presented in figure 1.

On the other hand, every logistic support system, operating under an increasingly complex and diverse system environment, may fail what, in consequence, may lead to:
- disruption of supporting task realization,
- inability of system to undertake a new task.

As a result, there is a need to take into account the possible unreliability of logistic support elements, which may lead to decrease of performance of the system being supported. On the background of these considerations, the analysis of operational system reliability or availability cannot be done in isolation without taking into account the numerous links with its logistic support system. The plethora of studies dealing with the problem of interactions between an operational system and its supporting systems, each of which are capable of independent operation arises when a need or a set of needs are met with a mix of multiple systems, and in fact develop the system of systems mostly base on the queuing theory. What means that the times to unit failure and unit repair have exponential probability distributions.

The table 1 serves to illustrate the examples of the above logistic support availability assessment models.

All the presented models investigate the problem of availability assessment for logistic support system and its supporting system separately. However, in order to achieve the availability assessment, there is a need both to integrate the logistic system with the operational system into one ‘system of systems’ model [23].

According to the definition, the system of systems context arises when a need or a set of needs are met with a mix of multiple systems, each of which are capable of independent operation but must interact with each other in order to provide a given capability. The loss of any part of the system will degrade the performance of the whole [5].

The literature on modeling system of systems is still scarce. The interactions between an operational system and its supporting system have not been clearly investigated. As a result, the major questions in this research area to be discussed are:
- how to describe the interactions between these two mentioned systems, and in fact develop the system of systems?
- what does it mean that system of systems is available?
2. System of systems with time dependency

In many systems undesired event occur later than components failure, if and only if the repair is not completed within a grace period. In other words, time redundant system has the ability to tolerate interruptions in their basic function for a specific period of time without having the negative impact on the system task performance.

Typically, the time redundant systems have a defined time resource, denoted by \( \gamma \) that is larger than the time needed to perform the system total task. However, unreliability of system element may cause time delays which in turn would cause the system total performance time to be unsatisfactory. As a result, considering the system task completion time as a random variable, the probability that mentioned time will be longer than the restricted time resource may be defined as the unreliability index \([13, 23]\).

According to the present knowledge, the time redundancy is considered as the effective tool for e.g. reliability improvement. In the case of two or more independent systems integration problem, time dependency is a convenient approach used to integrate these systems. After having analyzed the literature on modeling systems with time dependency (see e.g. \([19, 23]\)) it was possible to define the logistic support system for operational processes.

2.1. The model description

Consider a repairable system of systems under continuous monitoring, in which are integrated two independent systems: a single-unit operational system and its supporting system. Both systems have only two states: up state, when they are operable and can perform its specified functions, and down state, when they are inoperable.

Let’s assume that the operational system experiences random failures in time, and each failure entails a random duration of repair time before this system is put back into service. After repair mentioned system is ‘good-as-new’. Moreover, let’s also assume that any information about these failures is reliable and immediately comes to the logistic support system.

On the other hand, the logistic support functions are narrowed down to one main task – providing the necessary spare parts to the operational system. As a result, the logistic support system is inoperable when there is no capability of supplying the operational processes with necessary spares.

On the background of these considerations, in the logistic support area there can be especially used the individual time redundancy to model the system of systems performance \([23]\). Thus, if there is defined the system of systems total task as the continuous performing of exploitation process, the only way to provide it is the cooperation between the operational and its supporting system. As a result, the system of systems reliability is defined as its ability to correctly complete the task during the corresponding time resource \( \gamma \), which may be randomly distributed \([23]\).

Taking into account the above considerations, the probability that the system of systems at the random point in time is in up/downstate depends on:

- random variables describing the life times of the systems,
- random variable defining the repair time of operational system,
- random variable describing the delivery time of ordered spare parts,
- chosen stock policy, used in the logistic support system,
- restricted time resource.

### Tab. 1. The main models of logistic support systems for repairable items.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Type</th>
<th>System description</th>
<th>Methodology</th>
<th>No. of o. s. elements</th>
<th>No. of c/r.m.</th>
<th>Author(s)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Spare allocation models</td>
<td>three-echelon repairable item inventory system</td>
<td>Queuing theory</td>
<td>( J )</td>
<td>-</td>
<td>Coughlin (1984)</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>two- and three-echelon repairable item inventory system</td>
<td>Queuing theory/Markov processes</td>
<td>( N )</td>
<td>Gross, Gu &amp; Soland (1953)</td>
<td>finding the steady-state probability distribution of the Markov process</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Design models</td>
<td>aircraft spares provisioning decisions with respect to a user specified availability goals</td>
<td>Queuing theory</td>
<td>Ke &amp; Chu (2006)</td>
<td>finite queueing spare models connected with component redundancy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>redundant repairable system with one operating unit</td>
<td>Simulation</td>
<td>( 1 )</td>
<td>Kumar &amp; Sen (1995)</td>
<td>the use of the convolution of p.d.f. F of lifetime and d.f. G of repair time</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>single unit system supported by a single spare</td>
<td>Analytical</td>
<td>( 1 )</td>
<td>Sarkar &amp; Li (2006)</td>
<td>lifetime distribution – arbitrary continuous CDF with density function f, repair times are exponentially distributed</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>one-unit repairable system with s spares remained on cold standby</td>
<td>Analytical/numerical</td>
<td>( r )</td>
<td>Gunov &amp; Utkin (1995)</td>
<td>investigated different condition of repair</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>m-out-of-n redundant &amp; cold standby system</td>
<td>Analytical/numerical</td>
<td>( c )</td>
<td>Subramanian &amp; Natarajan (1981)</td>
<td>PM rate is a constant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>n-unit warm standby system with r repair facilities and PM</td>
<td>Queuing theory</td>
<td></td>
<td>Destombes, Heijden &amp; Harten (2004)</td>
<td>p.d.f. functions are exponentially distributed</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>k-out-of-n redundant &amp; hot standby system</td>
<td>Exact/Approximate method</td>
<td>( n )</td>
<td>Jain &amp; Maheshwari (2004)</td>
<td>system with reneging</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>machine repairable system with m operating units &amp; s warm standby units</td>
<td>Queuing theory/numerical</td>
<td>( 1 )</td>
<td></td>
<td>investigated cold and warm standby</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>r-out-of-n standby system</td>
<td>Markov processes/numerical</td>
<td>( r = 1, , \text{rm} = c &gt; 1 )</td>
<td>Barnon, Frostig &amp; Levikson (2006)</td>
<td>-</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 2 illustrates the system of systems model, when the critical inventory level (CIL) is used as a stock policy. According to the scheme, the operational system experiences random failure in time. Information is immediately sent to its logistic system. When there is available spare element in the remaining stock, the necessary one is sent to the operational system. In this situation, the time of supply task performance, denoted by \( \tau \), lasts from the moment of failure till the new delivery arrival. Finally the operational system is put back to service.

If there is restricted the system of systems total task completion time, defined as the time of operational system recovery process, the system of systems remains in upstate if this defined time will be shorter than time resource. Otherwise, the system of systems will fail and remain in downstate till the end of operational system maintenance process.

The chosen stock policy affects the possible periods of time without spare parts and the way the system of systems performs. The moment, when inventory level in logistic system achieves critical point is the impulse to place a new order. Before the new delivery arrival, the operational system can only use limited amount of spare elements taken from the remaining stock. Demand that is not immediately satisfied is backordered and filled when a new order arrives. After the delivery, new elements are used according to system demand until the stock level falls again to the critical point. The time between two orders placing defines a procurement cycle.

The main measure, which may define the presented system of systems failure, is the probability of its downtime caused by over crossing the defined time resource by the system total task performance.

2.2. System of systems downtime caused by over crossing the defined time resource

In a single cycle the system of systems may fail if:

- time of supply task performance lasts longer than the defined time resource (system of systems downtime includes the lead-time from the moment of over crossing the \( \gamma \), and the time of operational system recovery),
- time of operational system recovery lasts longer than the defined time resource (system of systems downtime encom-

passes the remaining repair time from the moment of \( \gamma \) over crossing).

In order to evaluate the mathematical model of system of systems downtime, it is necessary to define the following assumptions:

- time to failure of operational system is described by known probability density function (p.d.f.) \( F(t) \) with density \( f(t) \),
- operational system recovery is defined as fault component replacing by a spare element taken from remaining stock in the logistic support system,
- repair time is described by known p.d.f. \( G(t) \) with density \( g(t) \),
- at the moment \( t = 0 \) spares level achieves the critical point and the new cycle begin,
- critical inventory level is equal to \( s \) elements in a stock,
- during the one procurement cycle operational system may use \( Q \) elements which is the ordered delivery quantity,
- lead-time (period between moments when a new order is placed and when it is delivered) is random variable with known p.d.f. \( E(t) \) and density \( \varepsilon(t) \),
- time resource \( \gamma \) is random variable described by known p.d.f. \( \Phi(t) \) with density \( \phi(t) \).

To determine how long system of systems downtime caused by over crossing the defined time resource can last, it is necessary to evaluate the probability \( n(t) \) that the last allowable operating element failure will occur in \( \Delta t \) period during one procurement cycle. It can be calculated as the \( s+1 \)-fold convolution of function \( f(t) \) \([7]\):

\[
n(t) = f^{s+1}(t)
\]

where:
- \( f(t) \) = probability that operating element will fail and its replacement will be finished during the \( \Delta t \) period, derived as a convolution of functions \( f(t) \) and \( g(t) \):

\[
f_{f}(t) = \int_{0}^{t} f(x) g(t-x) dx
\]

where:
- \( f(t) \) = probability of operational system failure,
- \( g(t) \) = probability of its recovery.

Let’s now assume, that random variable \( \xi_1 \) defines the period of time which elapses between two points in the procurement cycle:

- the moment when the whole process of operational system recovery over crosses the restricted time resource (the moment of system of systems failure),

Fig. 2. Time dependent system of systems model with a single-unit operable system and CIL as a stock policy
system of systems downtime, due to replacement time over spare parts in the logistic system (wable spare parts needed in the recovery process (the following formula:

\[ T = \int_0^\infty \psi(t) \cdot \psi(t + \xi) dt \]

where: \( \xi \) = random variable which defines the period of system of systems downtime due to supply task performance time over crossing the time resource \( \chi \). \( \psi(t) \) = probability density function which defines the period of supply task performance, obtained from the following formula:

\[ \psi(t) = \int \phi(t) \cdot \psi(t + \xi) dt \]

where: \( \tau \) = random variable which defines the period of time between the operating unit failure and the new delivery physically execution.

According to the assumptions, the available systems fail during the procurement cycle \( Q-L \) times, the supply task performance time over crossing the time resource. \( \phi(t) = \psi(t) \) = probability density function of time resource \( \chi \). \( \psi(t) \) = probability density function which defines the period of supply task performance, obtained from the following formula:

\[ \psi(t) = \int \phi(t) \cdot \psi(t + \xi) dt \]

where: \( \tau \) = random variable which defines the period of time between the operating unit failure and the new delivery physically execution.

Function \( \psi(t) \) defines the period of time between the operating unit failure and the new delivery physically execution.

According to the assumptions, when the operational system fails during the procurement cycle \( Q-L \) times there will be allowable spare parts needed in the recovery process (\( \tau = 0 \)) and only one time there can be such a situation that there is no available spare parts in the logistic system (\( \tau = 0 \)). As a result, the possible system of systems downtime, due to replacement time over crossing the time resource, defined by the following formula:

\[ w(t) = \int \phi(t) \cdot g(t + \xi) dt \]

Consequently, the probability of any system of systems downtime during the procurement cycle may be defined as:

\[ b_i(t, \xi) = \frac{1}{Q} b_i(t, \xi) + \frac{Q-1}{Q} w(t, \xi) \]

### 3. System of systems availability

Availability is the measure of the degree to which a system is capable of operating under stated conditions of use and maintenance, at an unknown (random) point in time [15]. If we make an assumption, that all elements which work in the system are characterized by identical probability distribution of lifetime and renewal time, the availability function can be defined by the following formula [7]:

\[ A(t) = 1 - F(t) + \int_0^t [1 - F(t - x)] h(x) dx \]

where: \( F(t) \) = cumulative of system lifetime probability distribution, \( h(t) \) = process renewal density given by the following formula [7]:

\[ h(t) = \sum_{i=0}^n f_i(t) \]

In order to obtain the availability function assessment for the analyzed system of systems with time dependency, it is necessary to consider three sub periods occurring during one procurement cycle. First sub period encompasses the time between the moment the stock reaches CIL and the moment the last allowable spare part is used. During this sub period availability depends on:

- time to failure of s allowable elements,
- time of operational system recovery.

The second sub period encompasses the time to failure of \( s+1 \) element and its replacement time. In this sub period the system of systems may fail due to lack of spare parts or/and too long replacement time. The third sub period includes the time between the moments when the new delivery arrives till the instant of time when the stock achieves again the ordering point. Availability depends on:

- time to failure and recovery time of operational system,
- time of supply task performance.

As a result, the renewal density function \( h(t) \) changes during the procurement cycle. When the new cycle begins (inventory level reaches CIL):

\[ h_i(t) = \sum_{j=0}^n f_i(j) \]

where: \( f_i(t) \) = convolution of functions \( m(t) \) and \( w(t) \):

\[ f_i(t) = \int m(t - x) w(x) dx \]

where: \( m(t) \) = probability density function which defines the system downtime due to replacement time over crossing the time resource, \( m(t) \) = probability that a system of systems will fail in the \( \Delta t \) period, given by the formula:

\[ m(t) = \int_{t_0}^{t_0} f(t - x) \rho(x) dx \]

When all allowable spare elements are used, system of systems may fail also because of supply task performance time over crossing the time resource. Thus, the sub renewal density is expressed by the formula:

\[ h_i(t) = \int [m(t - x) + h_i(x)] dx \]

As a result, all moments of consecutive failures may shift in time, and the sub renewal density may be defined as:

\[ h_i(t) = \sum_{j=0}^n \int f_i(t - x) \rho(x) dx \]

Consequently, the process renewal density is expressed as:

\[ h(t) = h_i(t) + h_i(t) + h_i(t) \]

The mathematical model of system availability function is almost never used in practice. Instead of it, there can be evaluated the steady-state availability ratio.

#### 3.1. System of systems availability ratio

The basic formula for steady-state availability ratio assessment is expressed as follows [7]:

\[ A = \frac{T^0}{T^0 + T^e} = 1 - \frac{T^e}{T^0 + T^e} \]

where: \( T^0 \) = expected system’s time to failure, \( T^e \) = expected repair time.
For the presented system of systems with time dependency, the mean availability in one procurement cycle is expressed as:

\[
A = 1 - \frac{T^S}{Q(T^O + T^S) + T^T}
\]

where: \(T^S\) = expected system of systems downtime caused by the time of operational system recovery, \(T^O\) = expected time to failure of operational system, \(T^T\) = expected replacement time of operational system, \(T^R\) = expected supply task performance time, \(Q\) = ordered delivery quantity which is accessible to be used during a single cycle.

The expected values, which define the mean duration of systems uptimes and downtimes in one procurement cycle, may be obtained from the following formulae:

- expected time to failure of operational system:
  \[
  T^O = \int_0^\infty f^O(t) \, dt
  \]
  (18)

- expected time of operational system replacement:
  \[
  T^R = \int_0^\infty g^O(t) \, dt
  \]
  (19)

- expected time of supply task performance:
  \[
  T^T = \int_0^\infty \psi(t) \, d\tau
  \]
  (20)

- expected system of systems downtime caused by system total task performance time over crossing the defined time resource:
  \[
  T^T = \int_0^\infty \psi(t) \, d\tau
  \]
  (20)

4. System of systems steady-state availability ratio when all distributions are to be exponential

To evaluate the availability function \(A(t)\) given by formulae (8)-(15), there have to be solved \(n\)-fold convolution of given functions (e.g. \(f^O(t), g^O(t), f(t)*g(t))\). According to the literature, there have been made some suggestions to employ the Laplace transform technique (see, for example, [7]). However, a lot of problems arise in inverting the Laplace transform. Except in the case when the underlying distributions are exponential, this is a formidable task [12].

According to the above considerations, there have been made following simplified assumptions to evaluate system of systems mean availability ratio:

- time to failure of operational system and its replacement time are exponentially distributed with hazard rate \(\lambda\) and repair rate \(\mu\),
- lead-time is random and its probability distribution is exponential with parameter \(\beta\),
- time resource is random and exponentially distributed with rate \(\nu\),
- system of systems is in steady state,
- the results from theoretical model are derived for quantity of \(s\) redundant elements kept as CIL, when \(s = 0, 1, 2, 3, \ldots\)
- relation: \(\frac{\lambda}{\mu} \leq 1\).

The level of system of systems steady-state availability ratio depends upon the particular level of hazard rates \(\lambda, \beta, \mu, \nu\) and quantity of spare parts kept as CIL. Obtained chosen results of sensitivity analysis for theoretical model of mean availability ratio are presented in figures: 3-6.

The influence of mean operational system lifetime on the availability ratio is characterized by the level of hazard rate \(\lambda\) (figure 3). As might be expected, the greater the time between failures of this system, the less is required expensive maintenance, critical test equipment, unique training, as well as other logistic elements. Moreover, the availability ratio increases. It is also worth mentioning, that when the time between failures is longer, there is more possible, that new delivery will arrive before all available spare parts are used in the recovery process. As a result, when CIL level increases, the availability ratio also increases.

The next example (figure 4) serves to illustrate the influence of mean lead-time on availability ratio. When system of systems is in steady-state and there are no supply deliveries performed (\(\beta = 0\)), availability ratio is equal to zero. On the other hand, when the time of delivery decreases (\(\beta\) increases), the probability that there will be no free spare parts when needed is lower. As a result, the probability that system of systems fails decreases what has the positive impact on the level of availability ratio.

As might be expected, there is also a strong connection between the level of mean time resource \(\gamma\), characterized by \(\nu\), and the availability ratio. The greater the level of time resource for described total task performance, the greater system of systems ability to tolerate any interruptions during its normal processes realization. As a result, the availability ratio also increases due to shorter system of systems downtime periods.

Finally, there can be analyzed the interaction between mean operational system downtime (characterized by \(\mu\)) and the level of availability ratio. When taking into considerations system of systems with \(s > 0\), the longer the system is inoperable, the greater the probability that system of systems fails. In consequence, the
availability ratio decreases. However, for those specific assumptions made for system of systems model development, taking into account system with s = 0 the availability ratio decreases despite shorter downtime periods. This kind of system of systems behavior is connected with supply process organization. The new order is placed when operational system fails. In that situation the time of supply task performance has great influence on availability ratio due to affecting the total time of operational system recovery process, and the probability of system of systems failure.

All the presented examples shows, that despite the level of parameters $\lambda, \beta, \nu, \mu$, the more spare parts is kept in the logistic supply system, the greater the level of availability ratio can be obtained.

Thus, the analysis results confirm the theoretical view of the relations between the rates and the availability ratio. The obtained results from the sensitivity analysis will be different for: e.g. other repair frequency, various mean repair time, various periods of lead-time, or other time resource $\gamma$.

5. Conclusions

For summarizing the above considerations, it has to be underlined that:

- the analysis of operational system availability cannot be done in isolation without taking into account the problem of support processes performance and its reliability,
- the presented theoretical system of systems model provides the convenient framework for defining the optimal time resource period taking into account the economical constraints,
- the complexity of integration problem between operational system and its supporting system confirms the necessity of further development.

6. References


